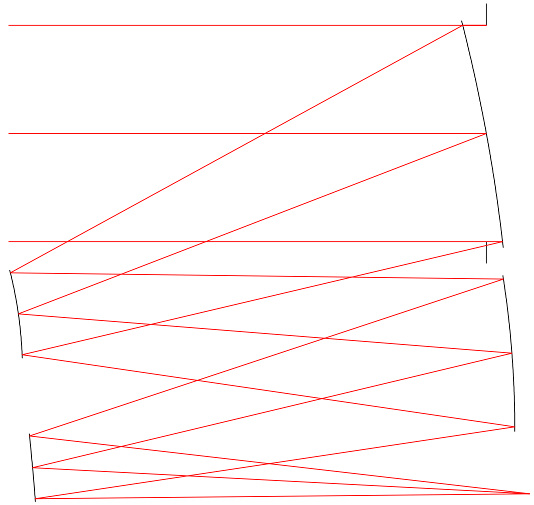


Four-Mirror Freeform Design

Jonathan C. Papa, Joseph M. Howard, and Jannick P. Rolland

NASA Technology Roadmap

2015 NASA Technology Roadmaps — **July 2015**
TA 8: Science Instruments, Observatories, and Sensor Systems

8.1 Remote Sensing Instruments and Sensors 8.1.3 Optical Components	**8.1.3.8 Wide Field of View Reflective Imager**

TECHNOLOGY

Technology Description: Allow the formation of an image on a flat detector to image near-Earth space from highly elliptical orbits.

Technology Challenge: Requires very clean facilities.

Technology State of the Art		Technology Performance Goal	
Technology State of the Art: Wide field-of-view (FOV) auroral imagers.		**Technology Performance Goal:** Develop fast wide FOV optics.	
Parameter, Value:	**TRL**	**Parameter, Value:**	**TRL**
FOV: 20 degrees; Aperture: 3 cm	9	FOV: 30 degrees; Aperture: > 60 cm; FOV: 5 degrees; Aperture: 200 cm	6

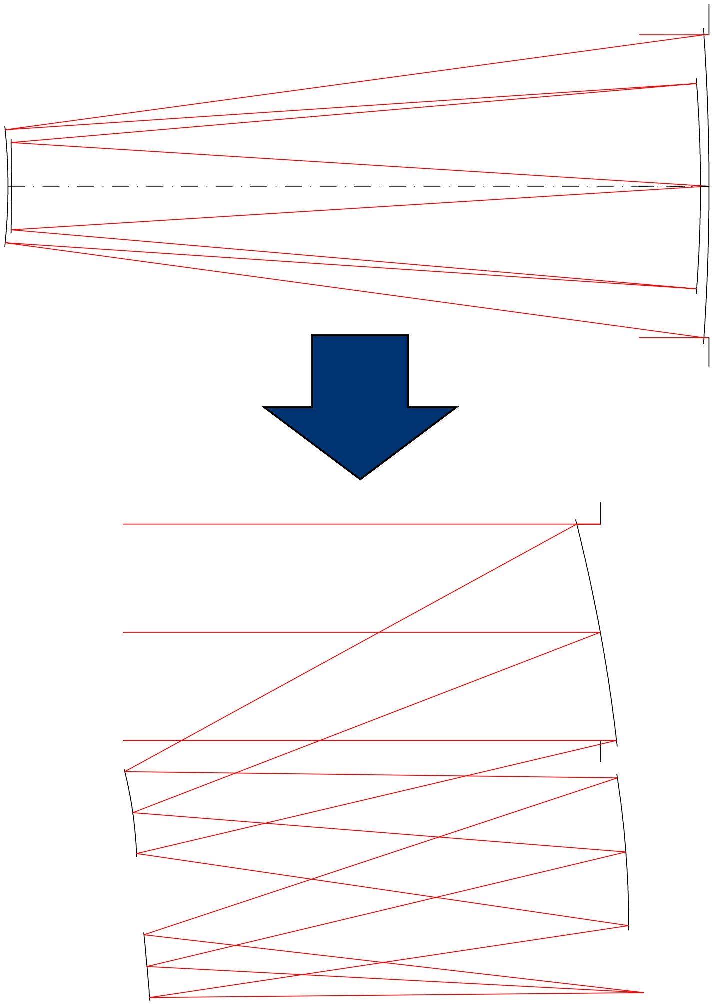
Technology Development Dependent Upon Basic Research or Other Technology Candidate: None

This work was supported by a NASA Space Technology Research Fellowship

Project Overview

- Central objective: survey of the four-mirror freeform solution space that considers geometries that could be advantageous for system constraints, such as mass, volume, stray light control, or radiation shielding.

- Methods/techniques: use analytically designed starting points before adding/varying freeform terms to explore different design forms.



Parts of the Design Process

Selection of Suitable Starting Point

Investigate multiple starting point designs.

Optimization Techniques

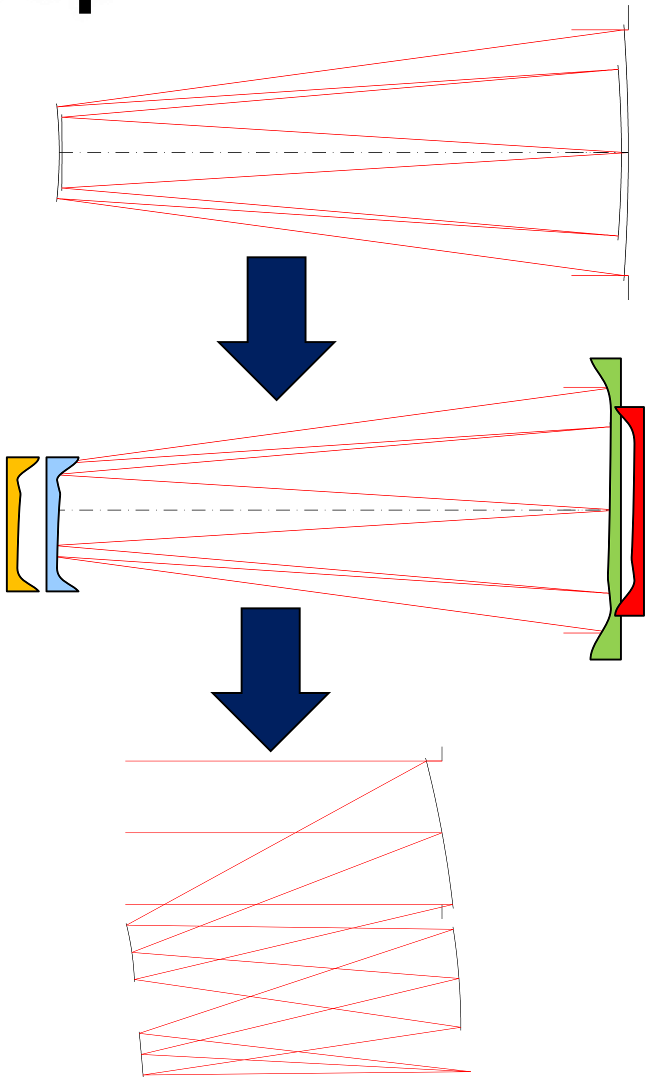
Investigate multiple optimization approaches.

Tolerancing/Sensitivity Analysis

Compare sensitivity across the different design forms.

Procedure Overview: Off-Axis Conics with Aspheric Caps on Top

1. Choose first order layout, and model as Cartesian reflectors, also known as confocal conics (If a flat-field is desired, mirror powers need to be balanced in this step).
2. Solve for 4th order aspheric terms on top of Cartesian reflector to correct third-order aberrations.
3. Choose tilts for the surfaces such that the system is unobscured and field-asymmetric field-linear astigmatism is canceled.



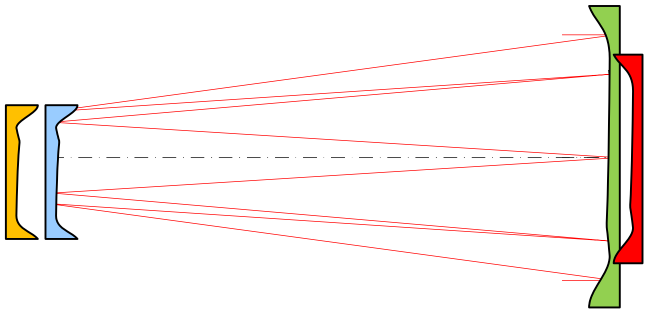
Key Points from Literature (Steps 1&2)

- Correcting Seidel third-order aberrations in a given first-order layout for 4 mirrors using aspheric deformations (aperture^4) is a linear system of equations [Korsch, 1973].
- A system made of Cartesian reflectors (conics with stigmatic imaging at each individual surface for the base field point, also known as confocal conics) can only be corrected through third-order if the system has a magnification of 1X, or if the system is afocal [Korsch, 1991].
- To make a focal system, with an infinite conjugate, that is corrected through third-order, you cannot use only Cartesian reflectors/confocal conics
  - For example: a classic Cassegrain with a stigmatically imaging parabolic primary and stigmatically imaging hyperbolic secondary, cannot correct for coma. A Ritchey-Chretien has spherical aberration introduced at the primary and canceled at the secondary in a way that cancels coma).

Korsch, Dietrich. "Closed-form Solutions for Imaging Systems, Corrected for Third-order Aberrations." Journal of the Optical Society of America 63.6 (1973): 667.

Korsch, D. Reflective Optics. Boston: Academic, 1991. Print.

Solving for Aspheric Caps

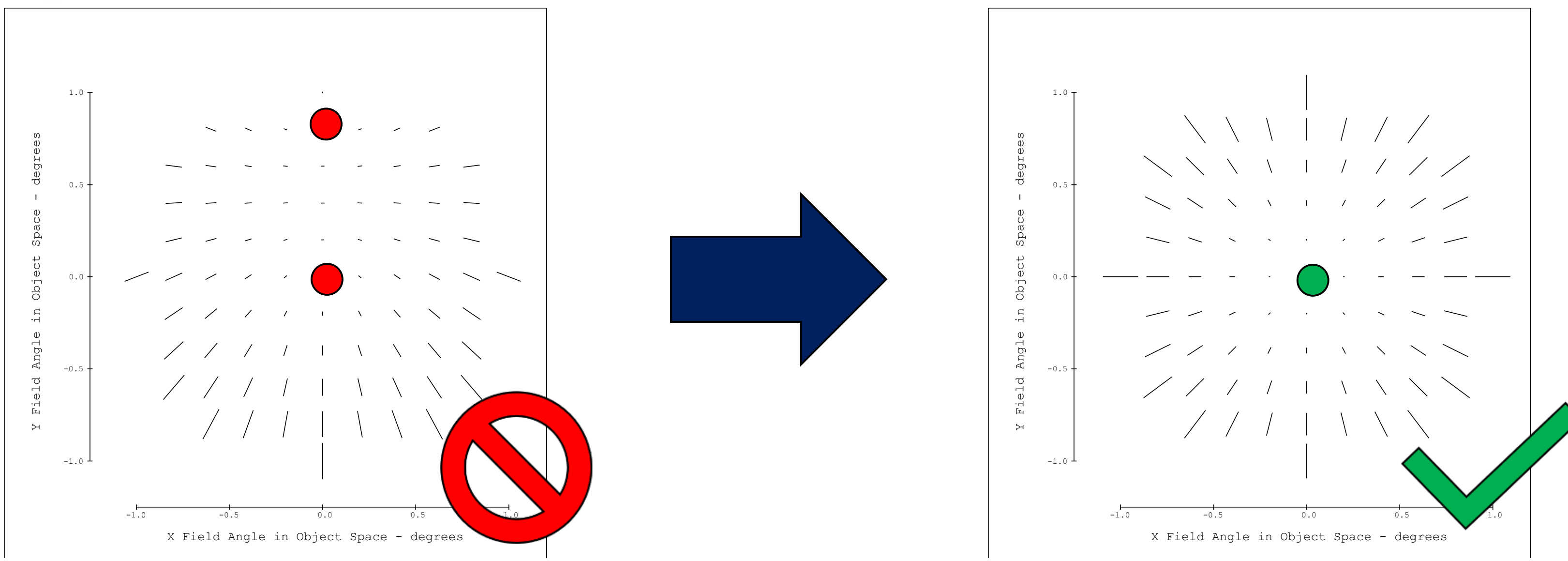


$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4\left(\frac{\bar{y}_1}{y_1}\right) & 4\left(\frac{\bar{y}_2}{y_2}\right) & 4\left(\frac{\bar{y}_3}{y_3}\right) & 4\left(\frac{\bar{y}_4}{y_4}\right) \\ 4\left(\frac{\bar{y}_1}{y_1}\right)^2 & 4\left(\frac{\bar{y}_2}{y_2}\right)^2 & 4\left(\frac{\bar{y}_3}{y_3}\right)^2 & 4\left(\frac{\bar{y}_4}{y_4}\right)^2 \\ 4\left(\frac{\bar{y}_1}{y_1}\right)^3 & 4\left(\frac{\bar{y}_2}{y_2}\right)^3 & 4\left(\frac{\bar{y}_3}{y_3}\right)^3 & 4\left(\frac{\bar{y}_4}{y_4}\right)^3 \end{bmatrix} \begin{bmatrix} \Delta W_{040_1} \\ \Delta W_{040_2} \\ \Delta W_{040_3} \\ \Delta W_{040_4} \end{bmatrix} = \begin{bmatrix} \Delta W_{040} \\ \Delta W_{131} \\ \Delta W_{222} \\ \Delta W_{311} \end{bmatrix} \begin{matrix} \text{Spherical} \\ \text{Coma} \\ \text{Astigmatism} \\ \text{Distortion} \end{matrix}$$

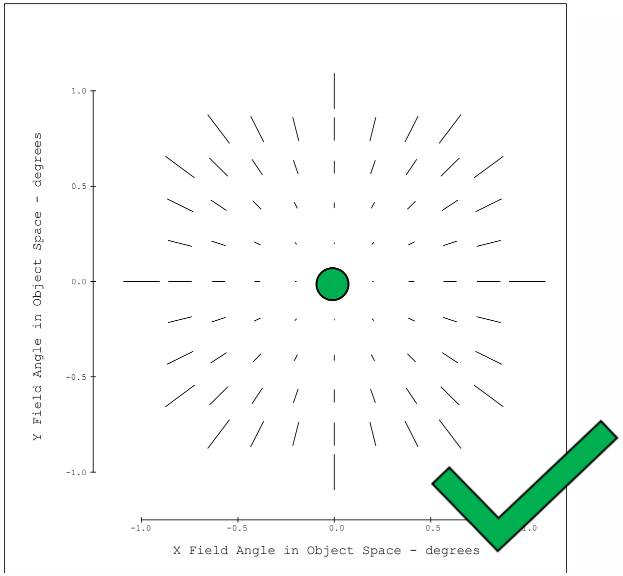
Related to Sag of 4th order aspheric caps

Key Points from Literature (Step 3)

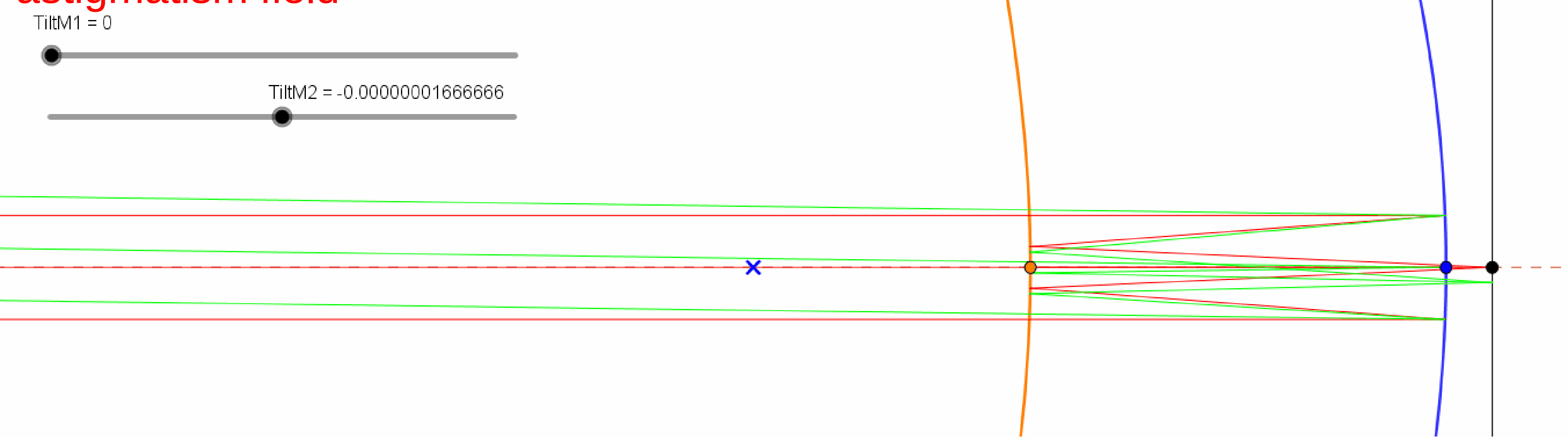
- It is possible to find tilted/decentered systems that exhibit aberrations of the “ordinary” kind (where two astigmatism nodes collapse to single node at center of field, while keeping the coma node at the center of field as well) [Rogers, 1986].
- Tilted aspheric term (looks like spherical aberration) produces aberrations of the “ordinary kind”, meaning field-linear coma, and field-quadratic astigmatism [Rogers, 1986].



Rogers, John R. "Vector Aberration Theory And The Design Of Off-Axis Systems." 1985 International Lens Design Conference (1986).



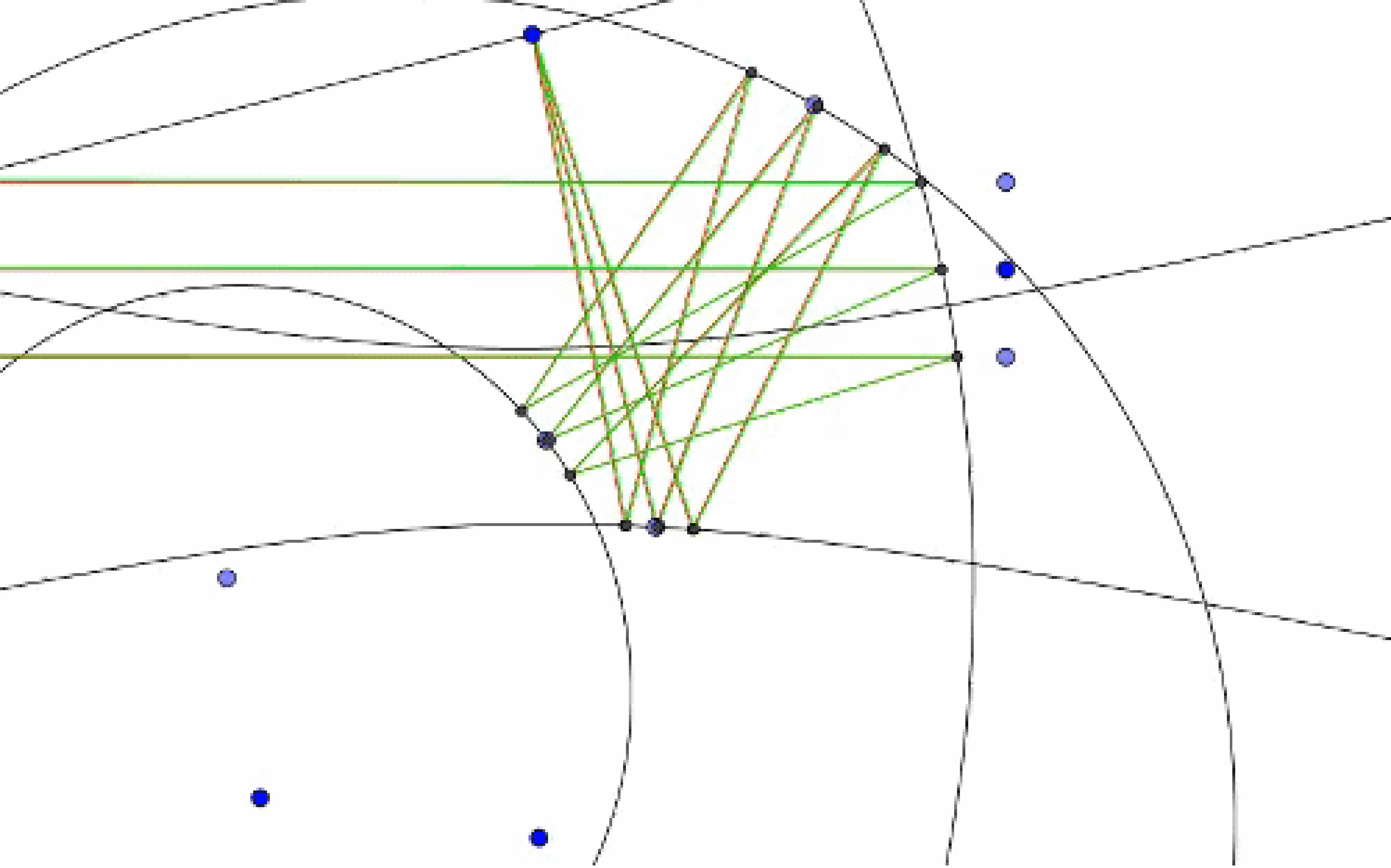
As the base ray is bent at M1, M2 bends to keep “ordinary” astigmatism field



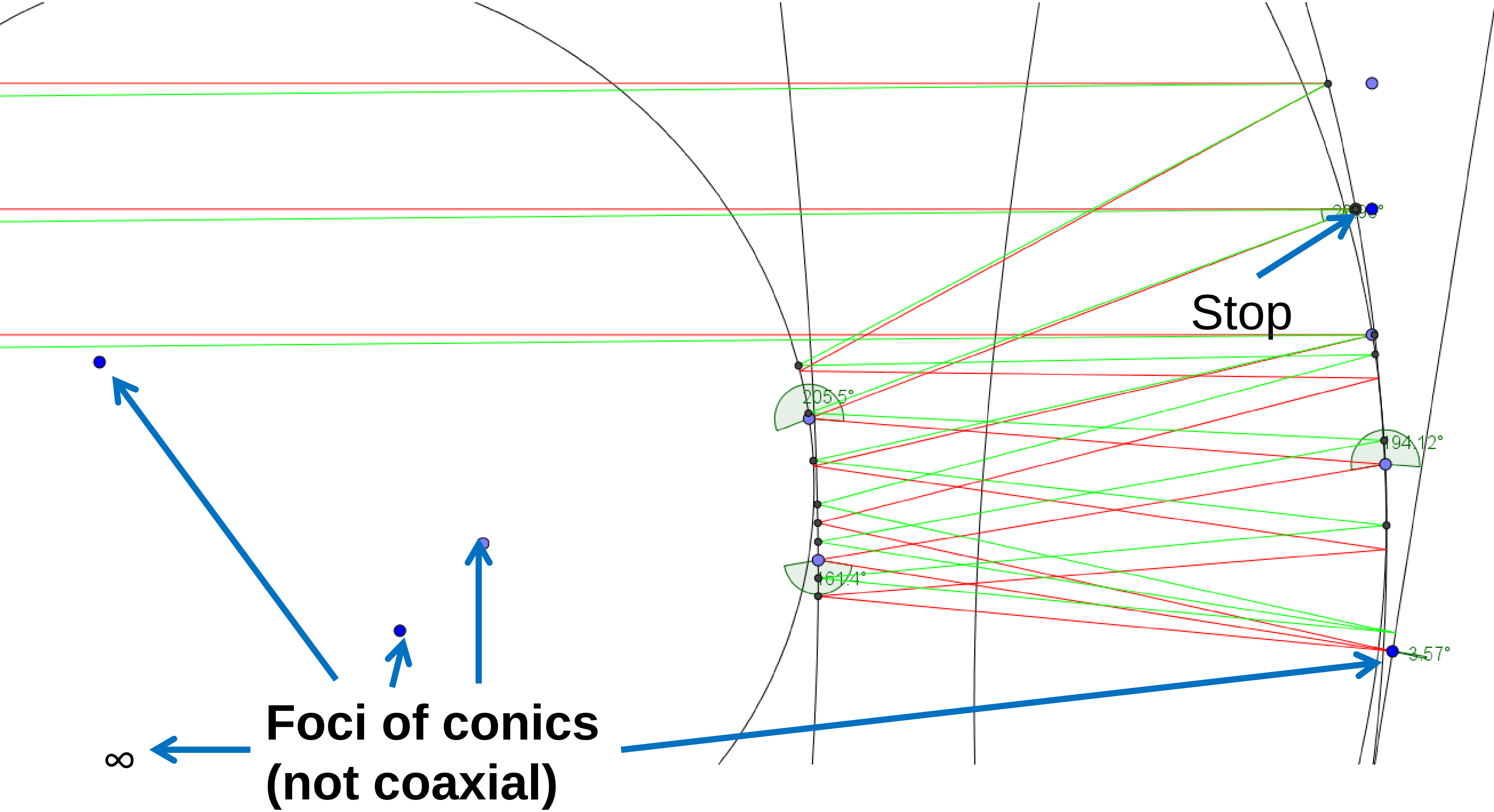
Key Points from Literature: Aberration Coefficients for Plane-Symmetric Systems

- Constraining a surface to be a stigmatically imaging off-axis conic eliminates [Sasian, 1994]:
  - Spherical Aberration
  - Field-Constant Coma
  - Field-Constant Astigmatism
  - Anamorphism
- Ignoring distortion and piston terms, the remaining image degrading aberrations are:
  - Field-Linear Coma
  - Field-Asymmetric Field-Linear Astigmatism (FAFL)
  - Field-Quadratic Astigmatism
  - Field Curvature
- The designer needs to eliminate FAFL Astigmatism to make the aberrations "ordinary" and correctable using aspheric caps.

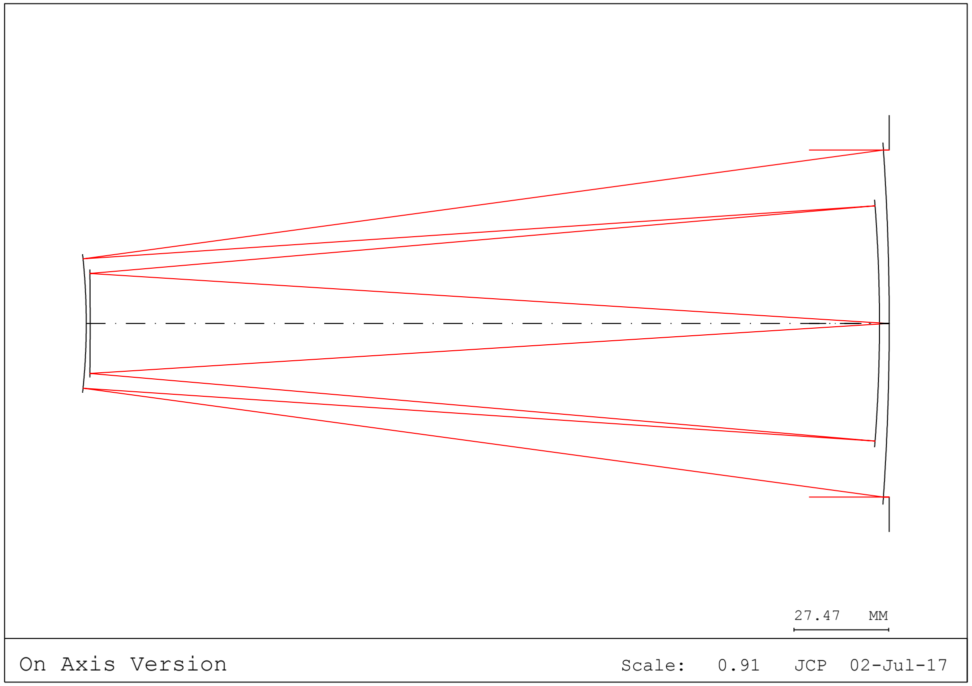
Sasian, Jose M. "How to Approach the Design of a Bilateral Symmetric Optical System."Optical Engineering 33.6 (1994): 2045.



Four-Mirror Example



Create On-Axis Equivalent, Third-Order Corrected

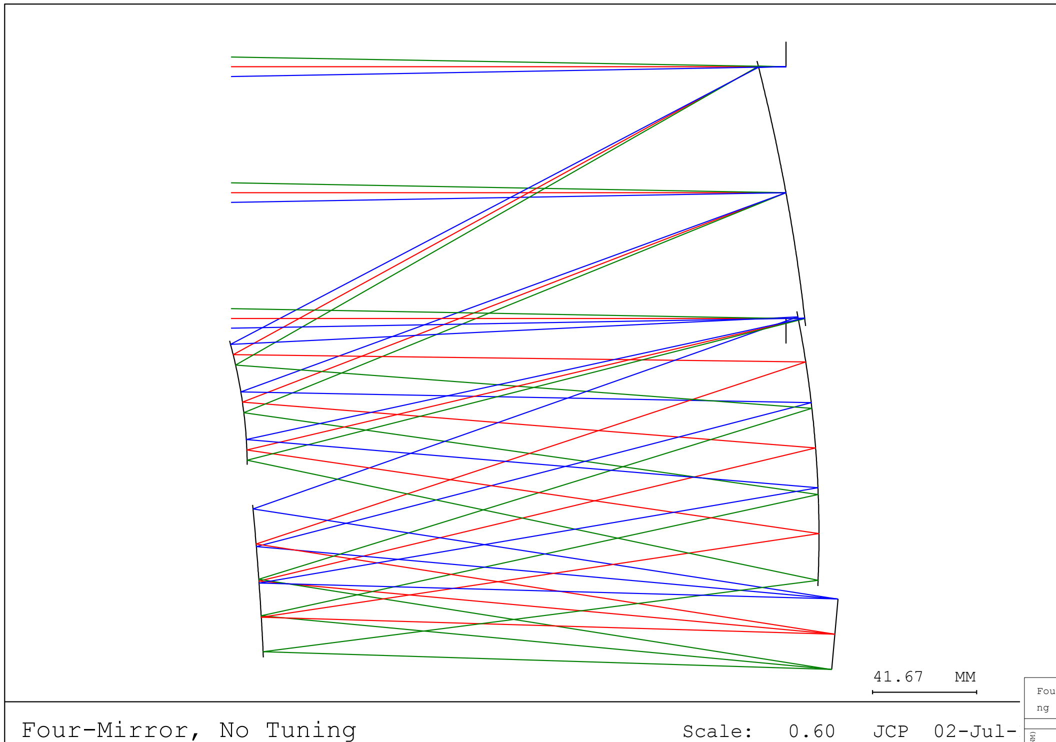


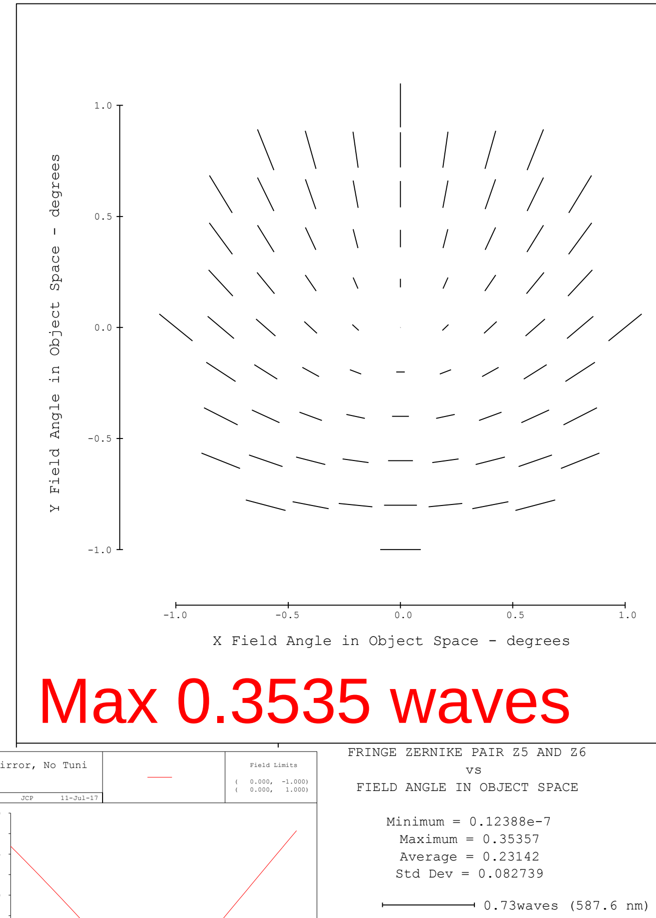
	M1	M2	M3	M4
Radius	-743.522	-186.402	-448.082	-1246.81
K	-1	-0.11681	-0.0161	-40.8171
A	6.46E-12	-5.05E-09	-7.06E-10	-6.38E-09

- K is the conic constant that corresponds to a stigmatic imaging conic for the axial field point.
- On top of these stigmatic conics is a 4th order "A" asphere coefficient that is used to correct the third-order field aberrations (field-linear coma, field-quadratic astigmatism) while leaving third-order spherical corrected.

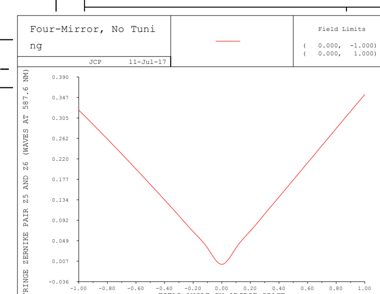
Save "Aspheric Caps" For Later Step

Create Chain of Stigmatic Imaging Conics (Aspheric Caps Removed), with Tilts That Cancel FAFL Astigmatism According to Aberration Coefficients



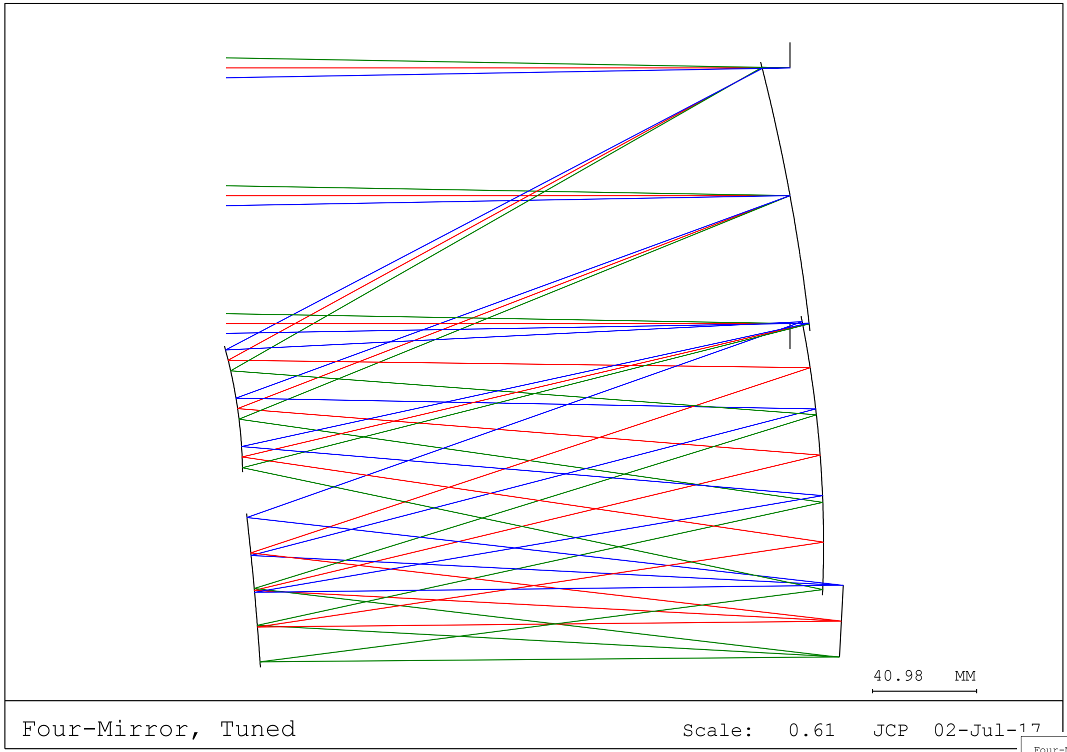


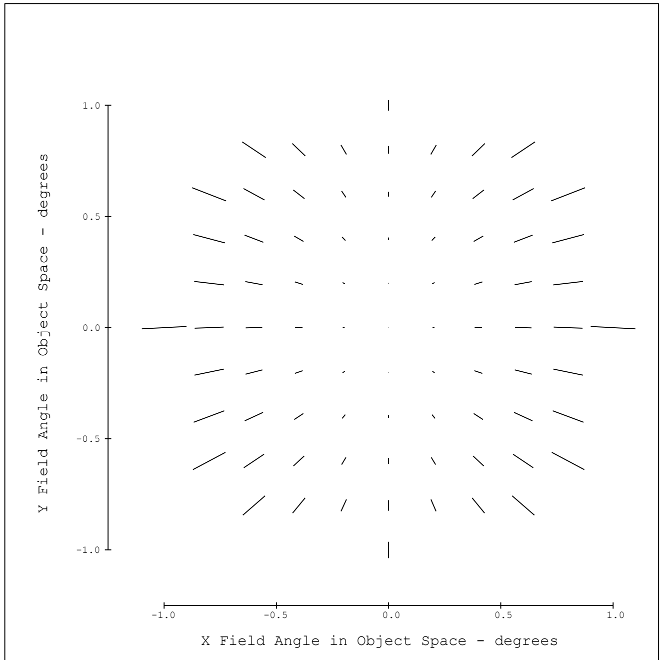
Max 0.3535 waves



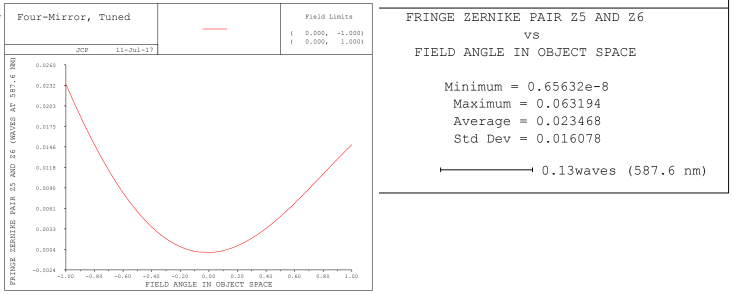
Astigmatism is not quite “ordinary”

“Tune” the Angle of the Last Mirror to Bring Astigmatism Nodes Together





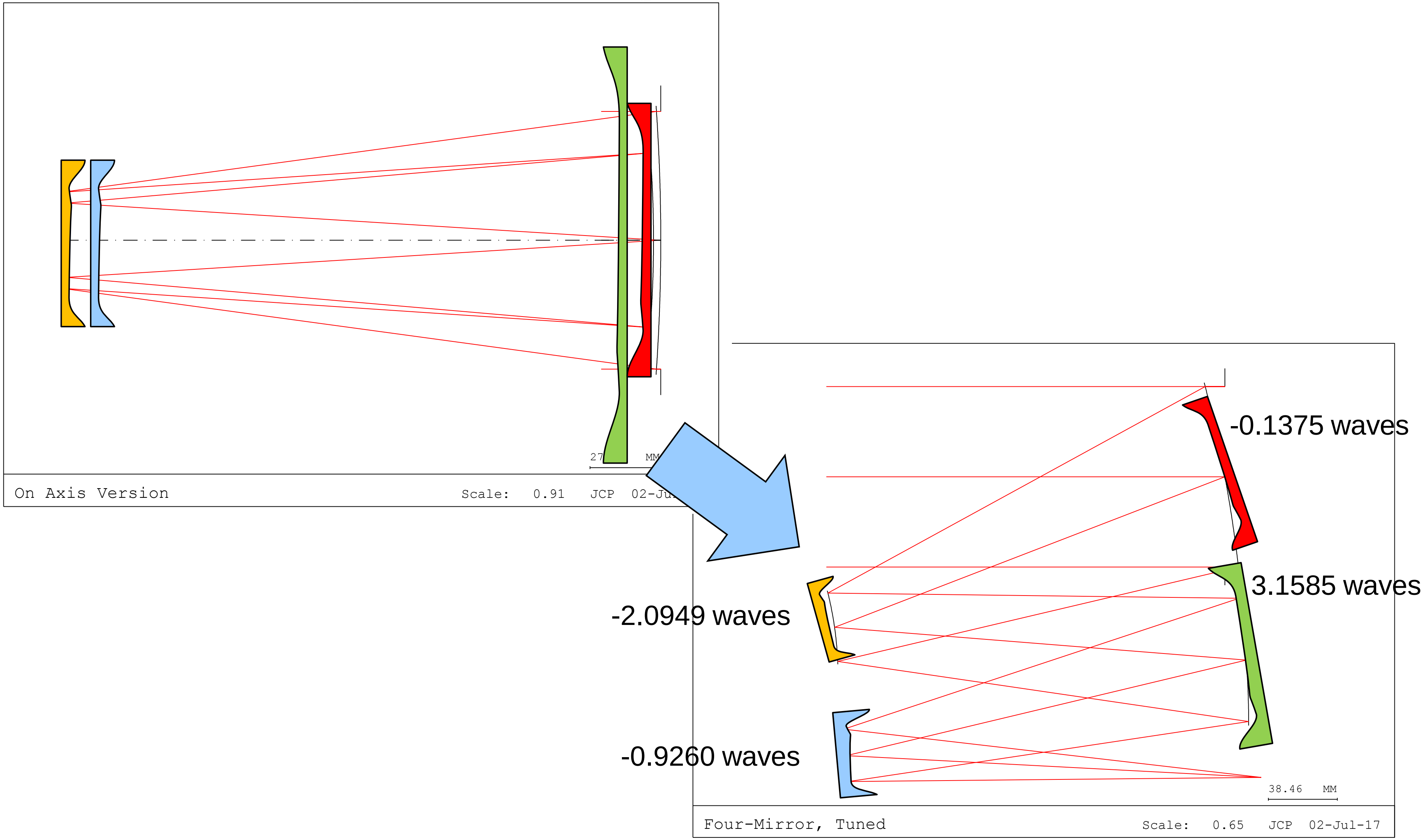
Max 0.063194 waves



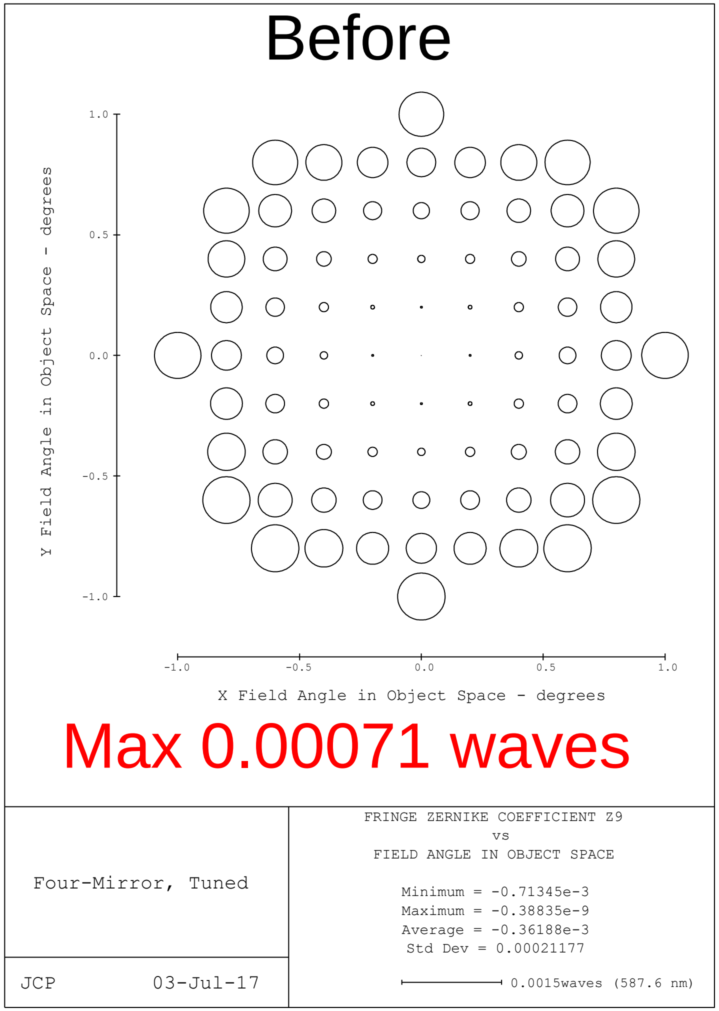
M4 tilt adjusted to -8.174° from -9.298°

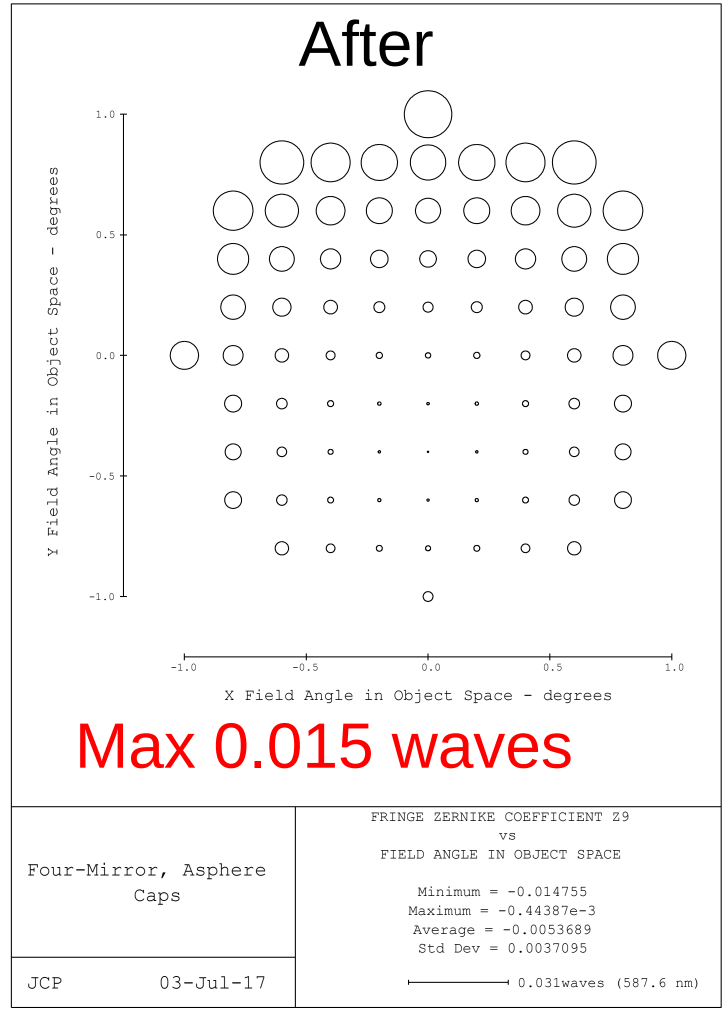
Astigmatism has single node, closer to “ordinary”

Put “Aspheric Caps” Back On

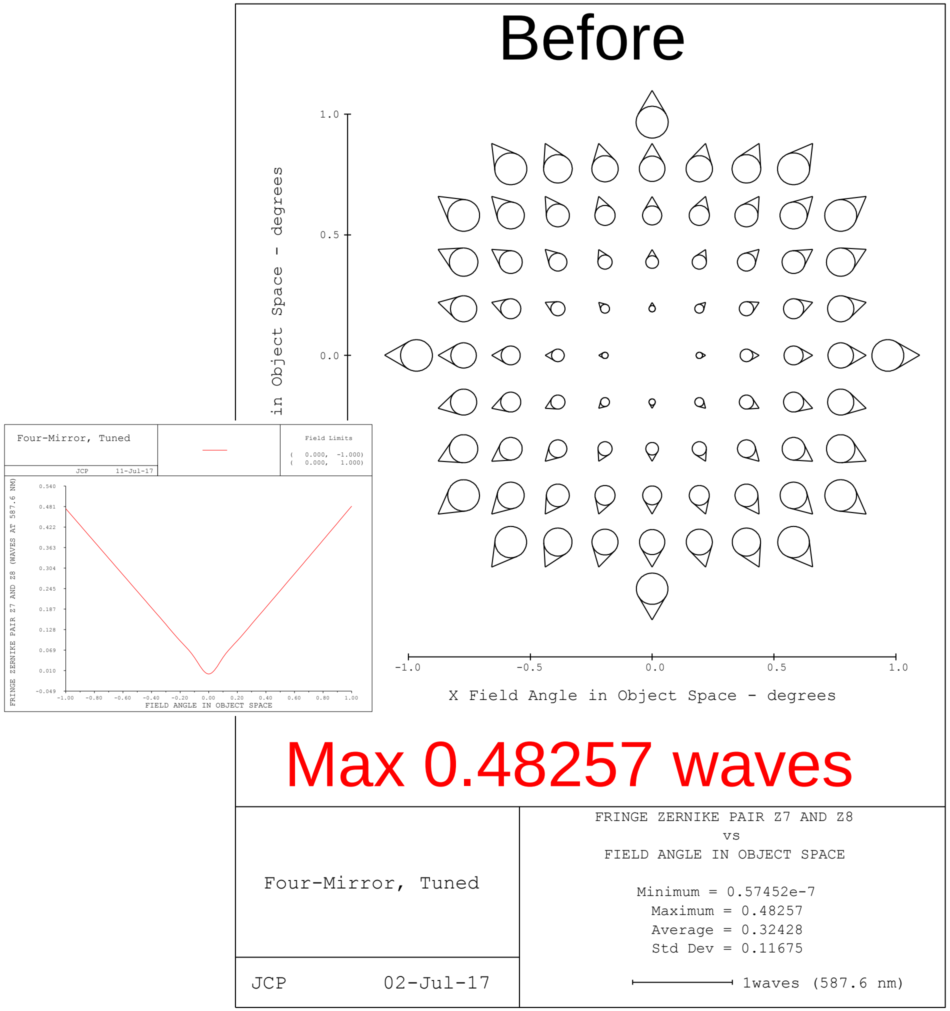


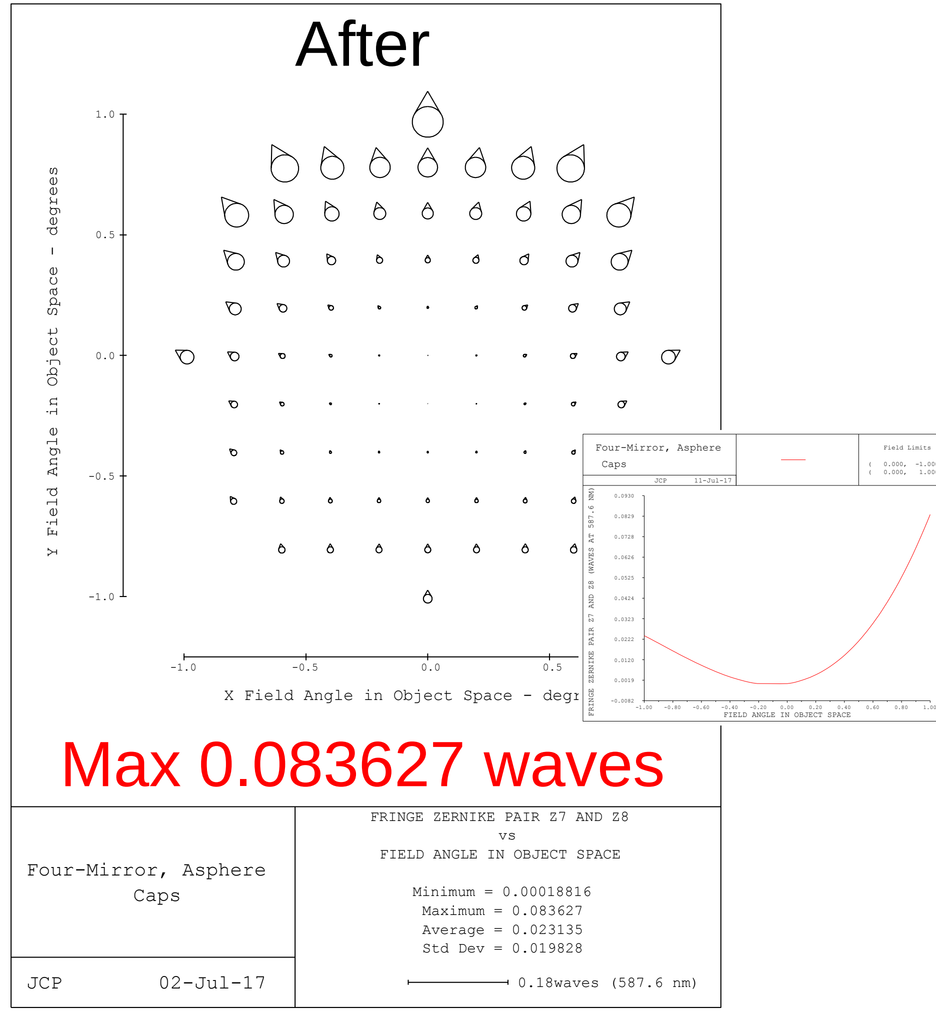
Aberrations Before and After Caps: Spherical Aberration





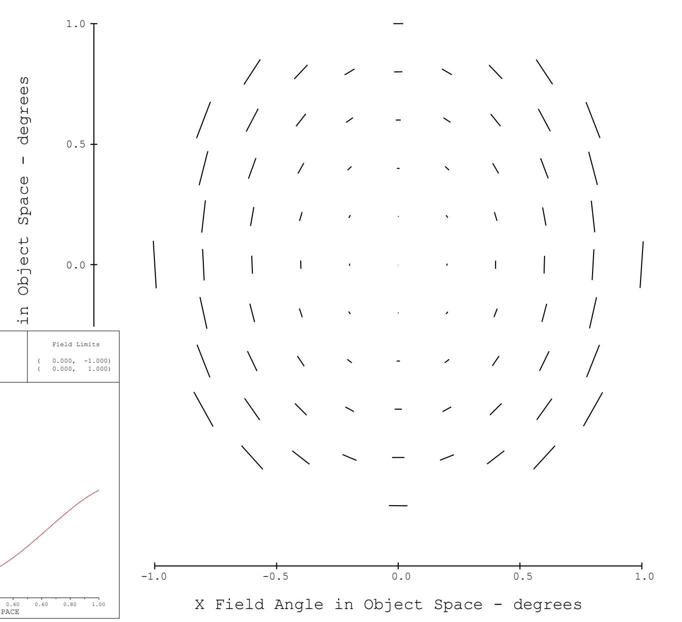
Aberrations Before and After Caps: Coma

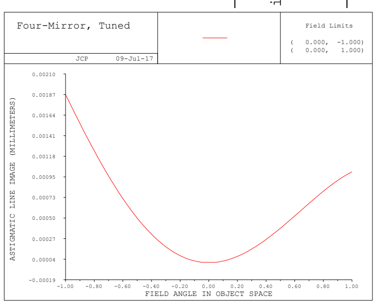




Aberrations Before and After Caps: Astigmatism (Coddington)

Before

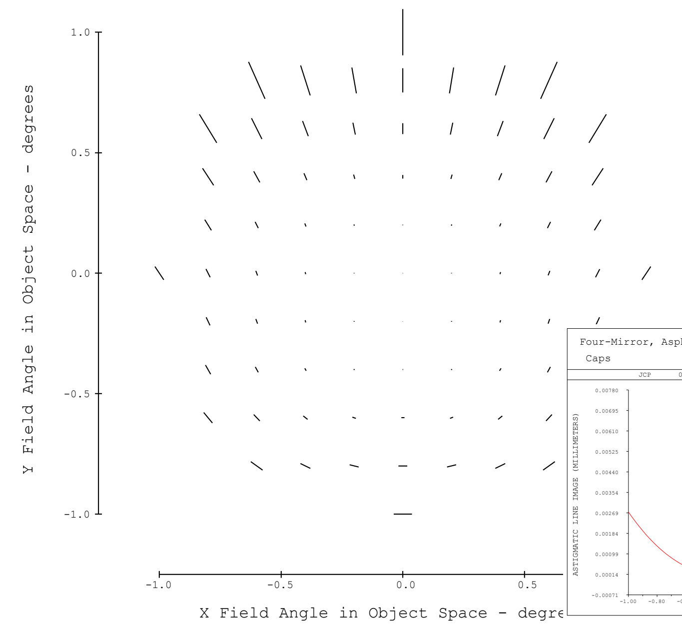


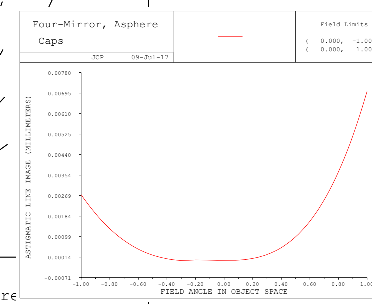


Avg. Line 1.81 microns

Four-Mirror, Tuned	ASTIGMATIC LINE IMAGE vs FIELD ANGLE IN OBJECT SPACE Minimum = 0 Maximum = 0.0048247 Average = 0.0018075 Std Dev = 0.0012178
JCP 02-Jul-17	0.01mm

After

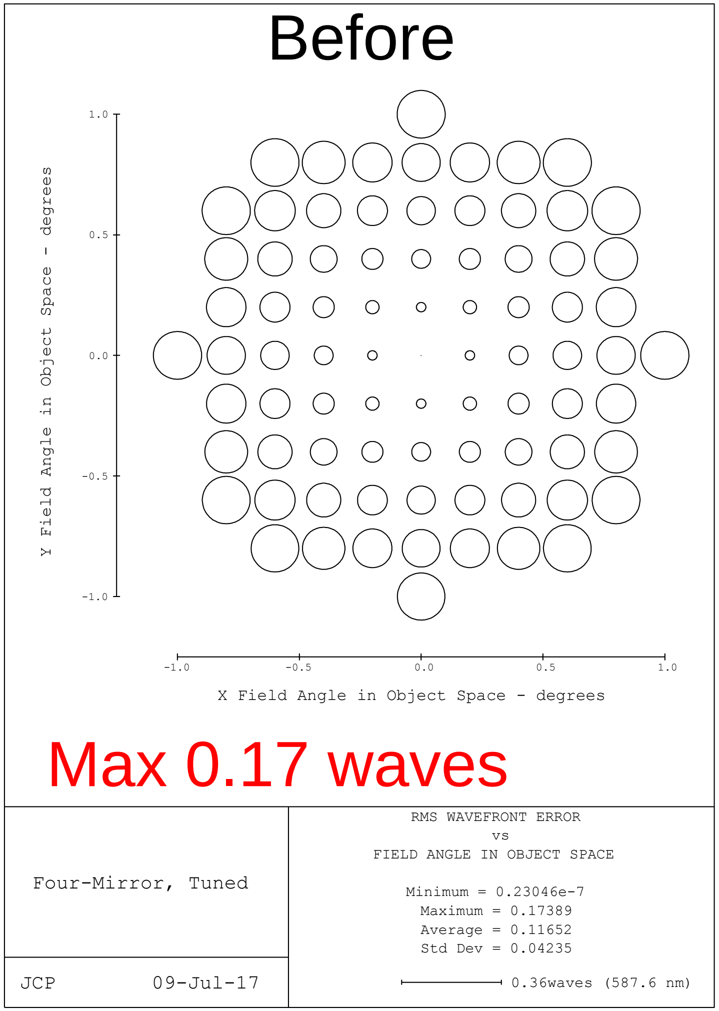


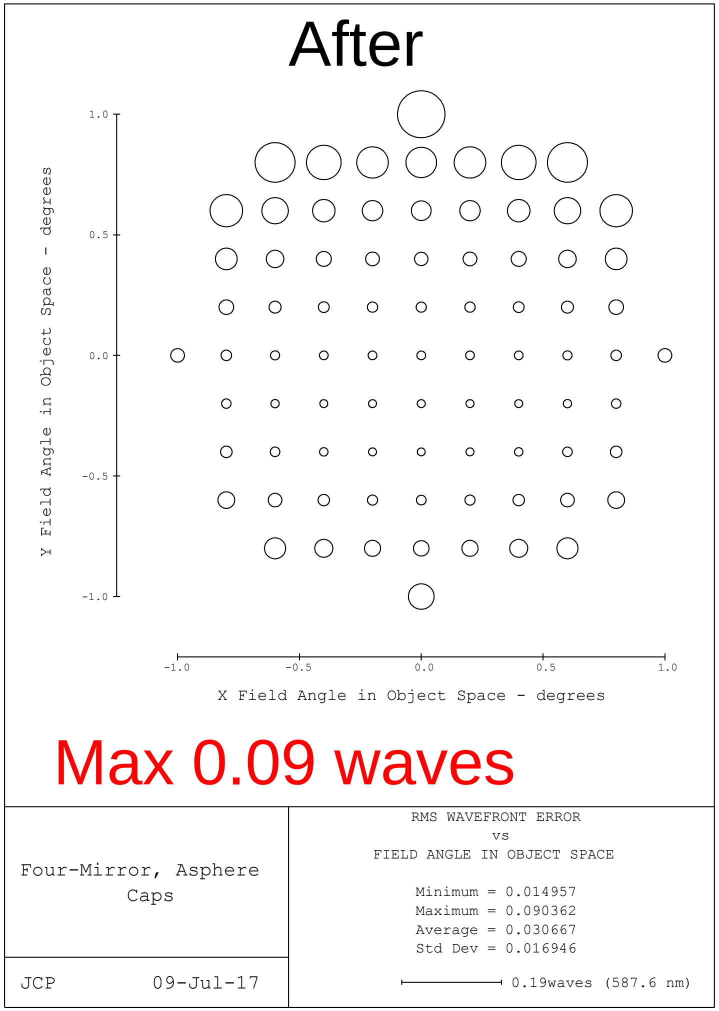


Avg. Line 1.67 microns

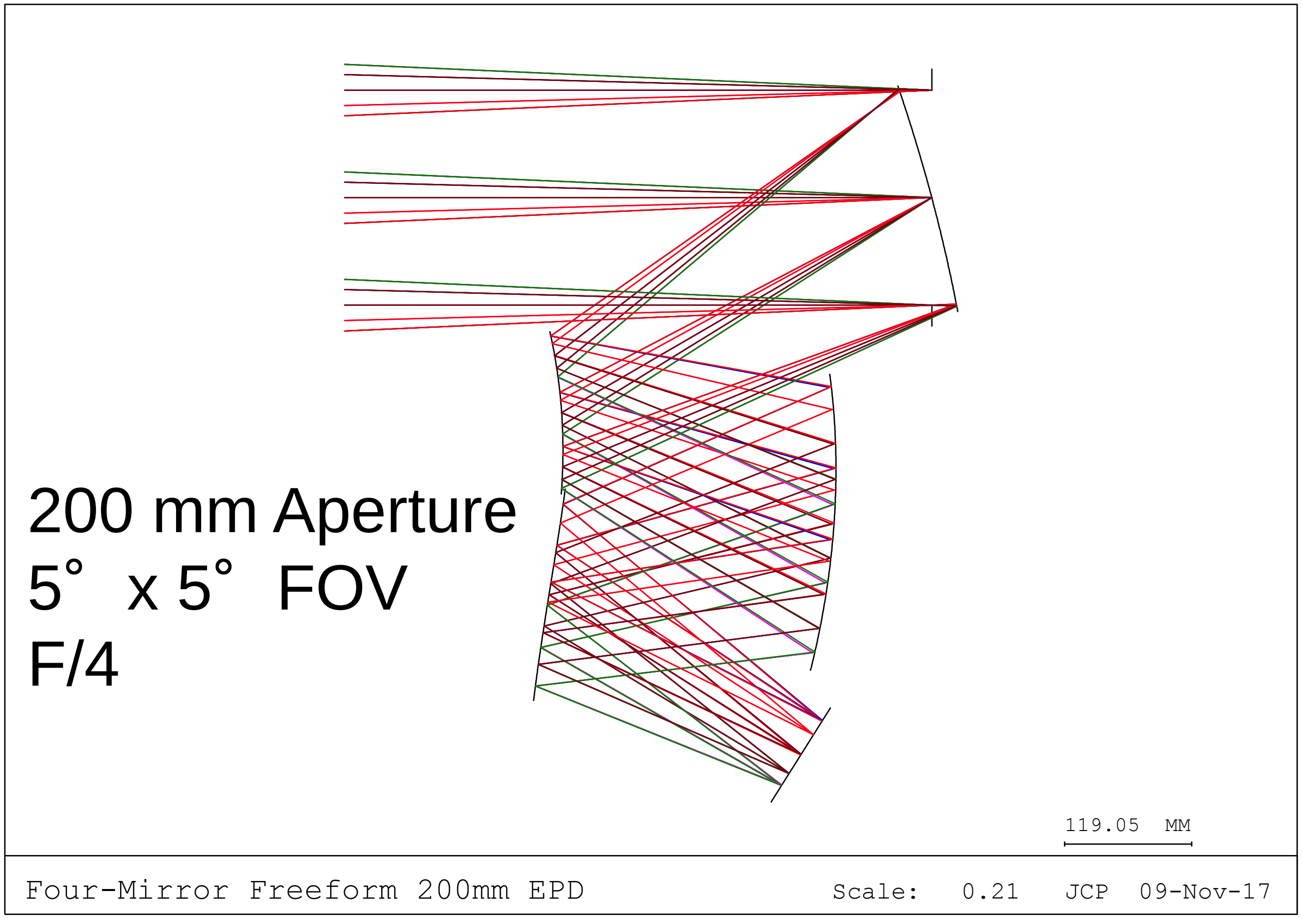
Four-Mirror, Asphere Caps	ASTIGMATIC LINE IMAGE vs FIELD ANGLE IN OBJECT SPACE Minimum = 0 Maximum = 0.0070252 Average = 0.0016681 Std Dev = 0.0015674
JCP 02-Jul-17	0.015mm

RMS Wavefront Error Before and After Caps





Approaching NASA Tech Challenge

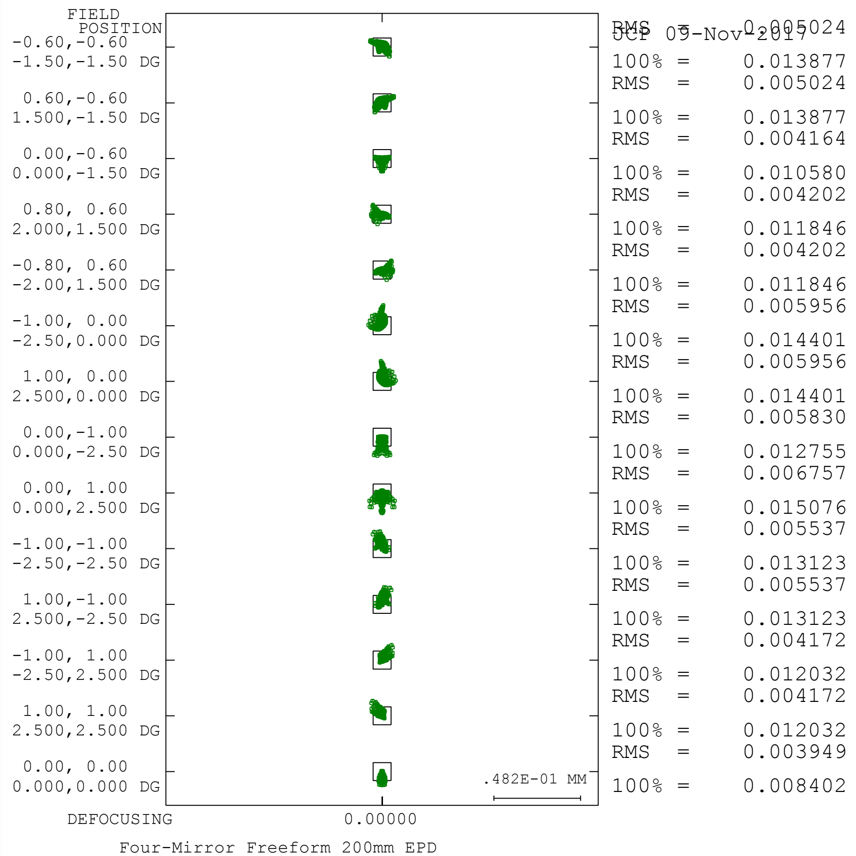


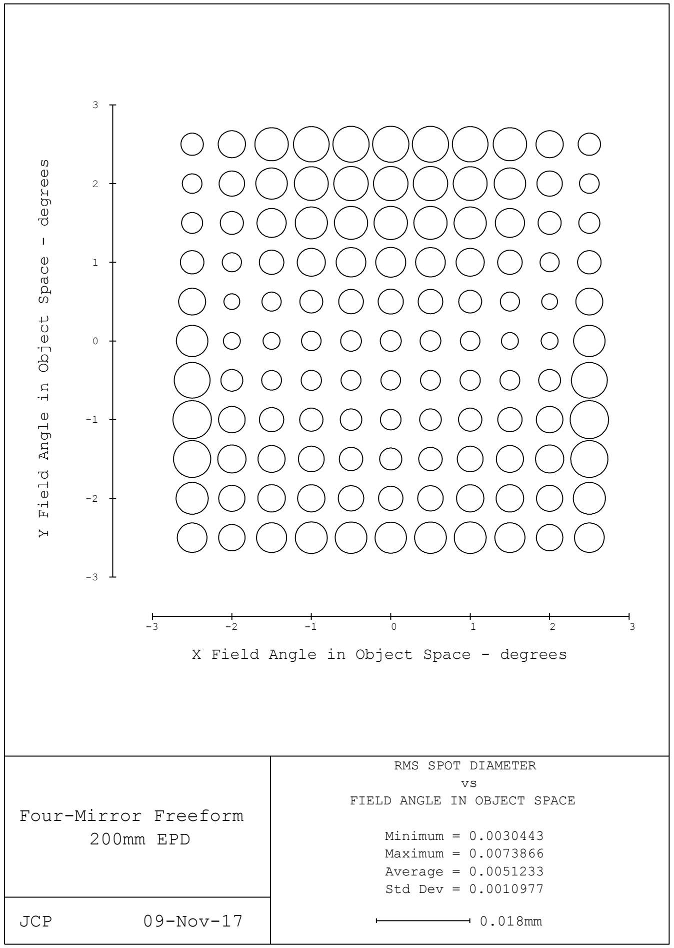
200 mm Aperture

5° x 5° FOV

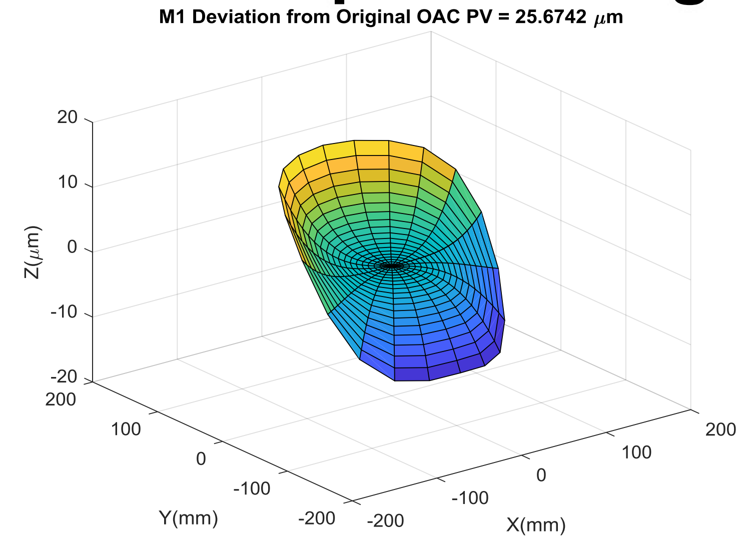
F/4

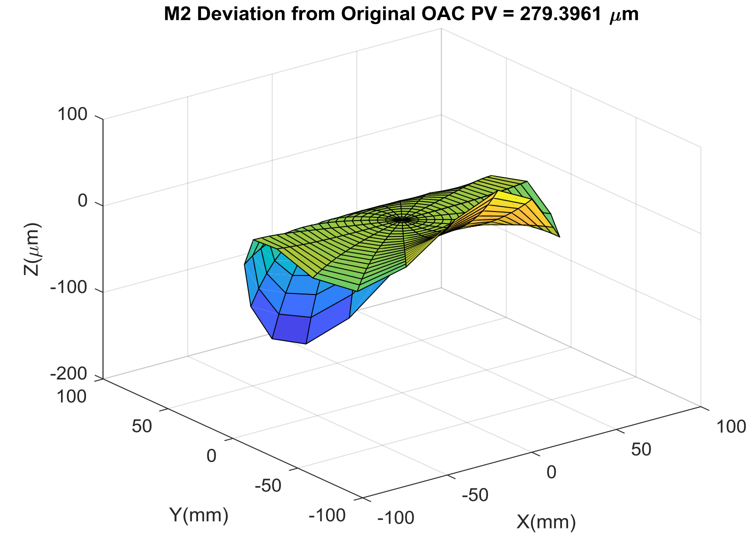
Four-Mirror Freeform Spot Diagram

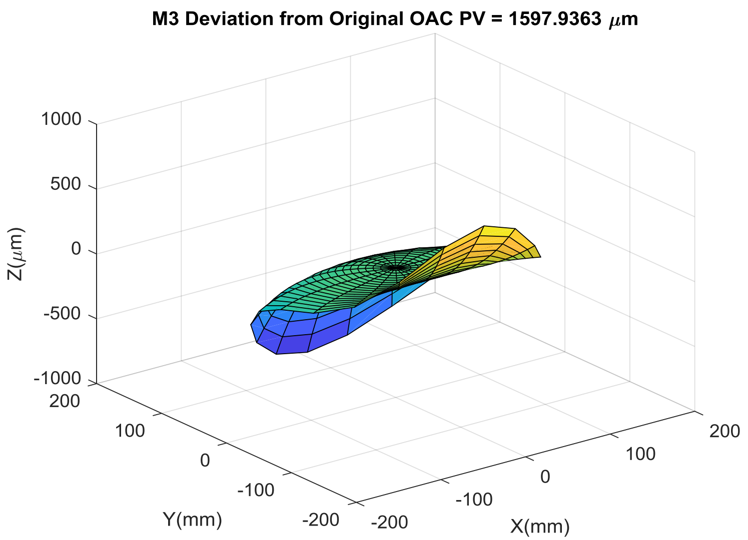


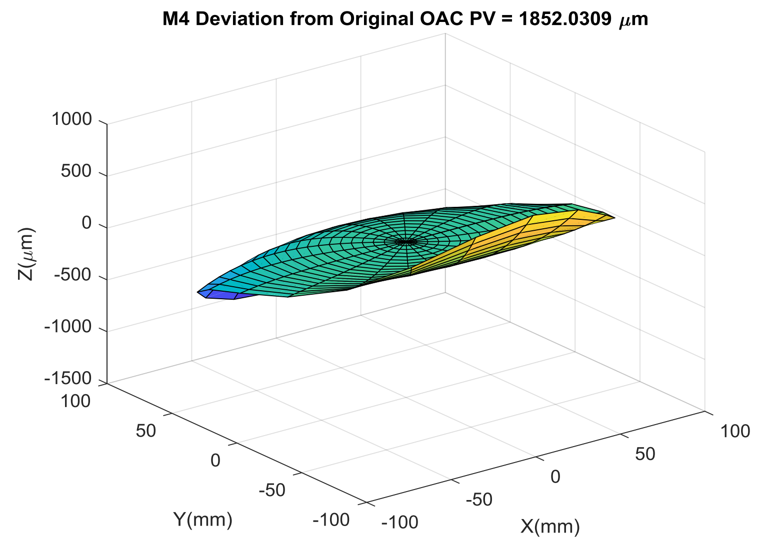


Freeform “Caps” on Original Base Off-Axis Conic

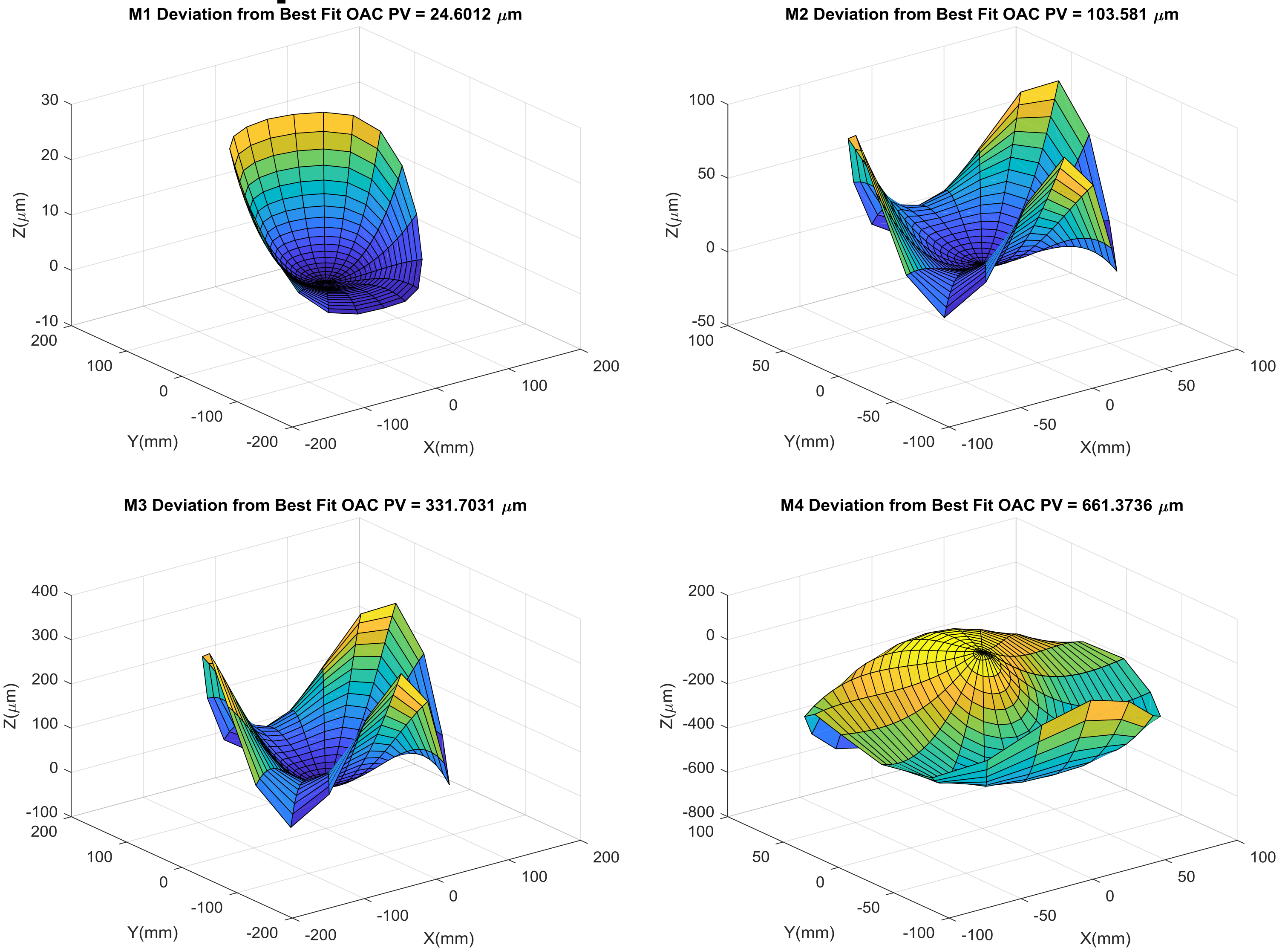








Freeform "Caps" on Refit Base Off-Axis Conic



Summary

- Several analytical starting point design methods exist with various symmetries and states of correction.
- A combination of these methods can allow for unobscured starting points that are corrected for third order image degrading aberrations, which will be used to facilitate a survey of the four-mirror freeform solution space.

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- Seunghyuk Chang, "Linear astigmatism of confocal off-axis reflective imaging systems with N-conic mirrors and its elimination," J. Opt. Soc. Am. A32, 852-859 (2015)

Put "Aspheric Caps" Back On

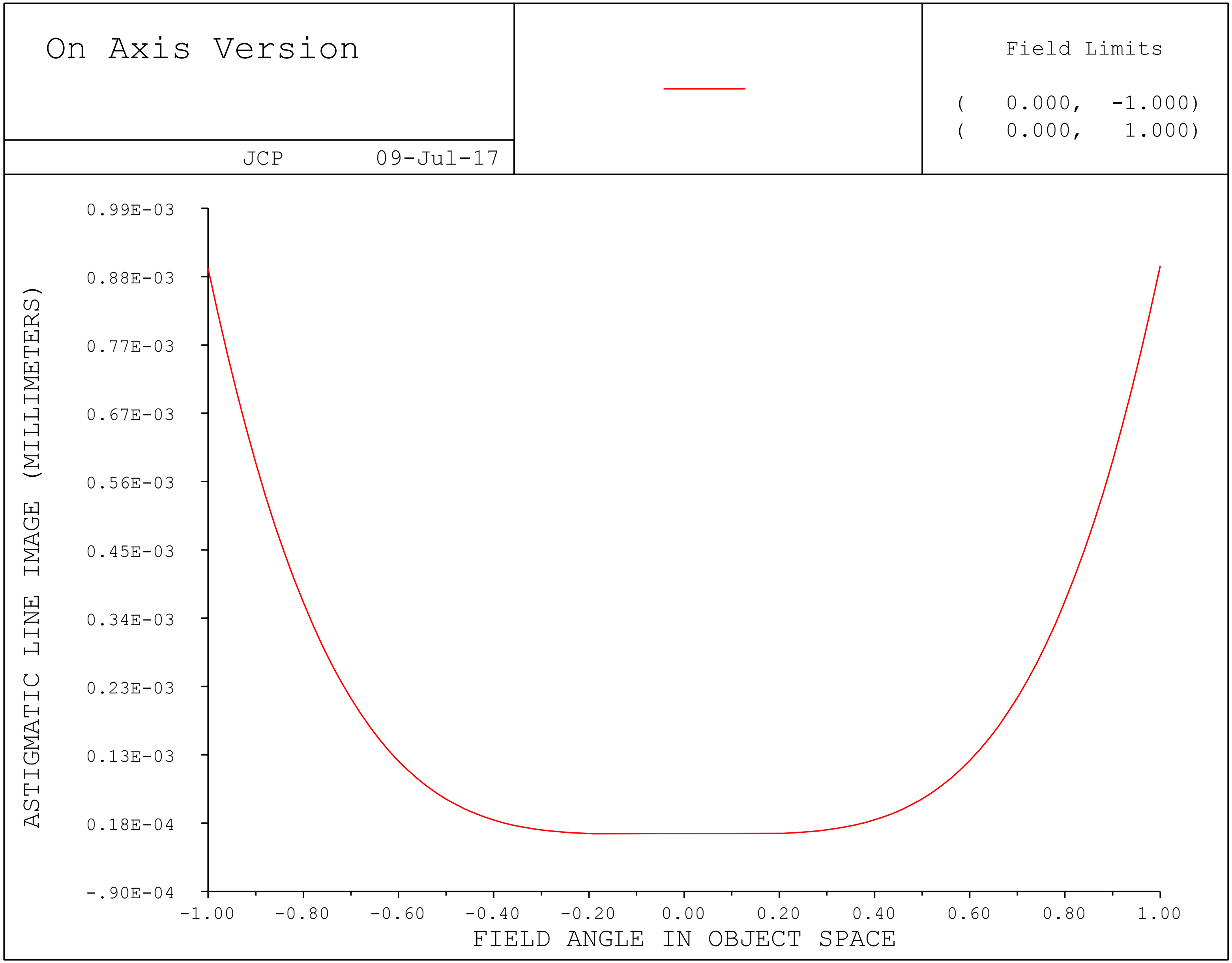
	M1	M2	M3	M4
Radius	-743.522	-186.402	-448.082	-1246.81
K	-1	-0.11681	-0.0161	-40.8171
A	6.46E-12	-5.05E-09	-7.06E-10	-6.38E-09

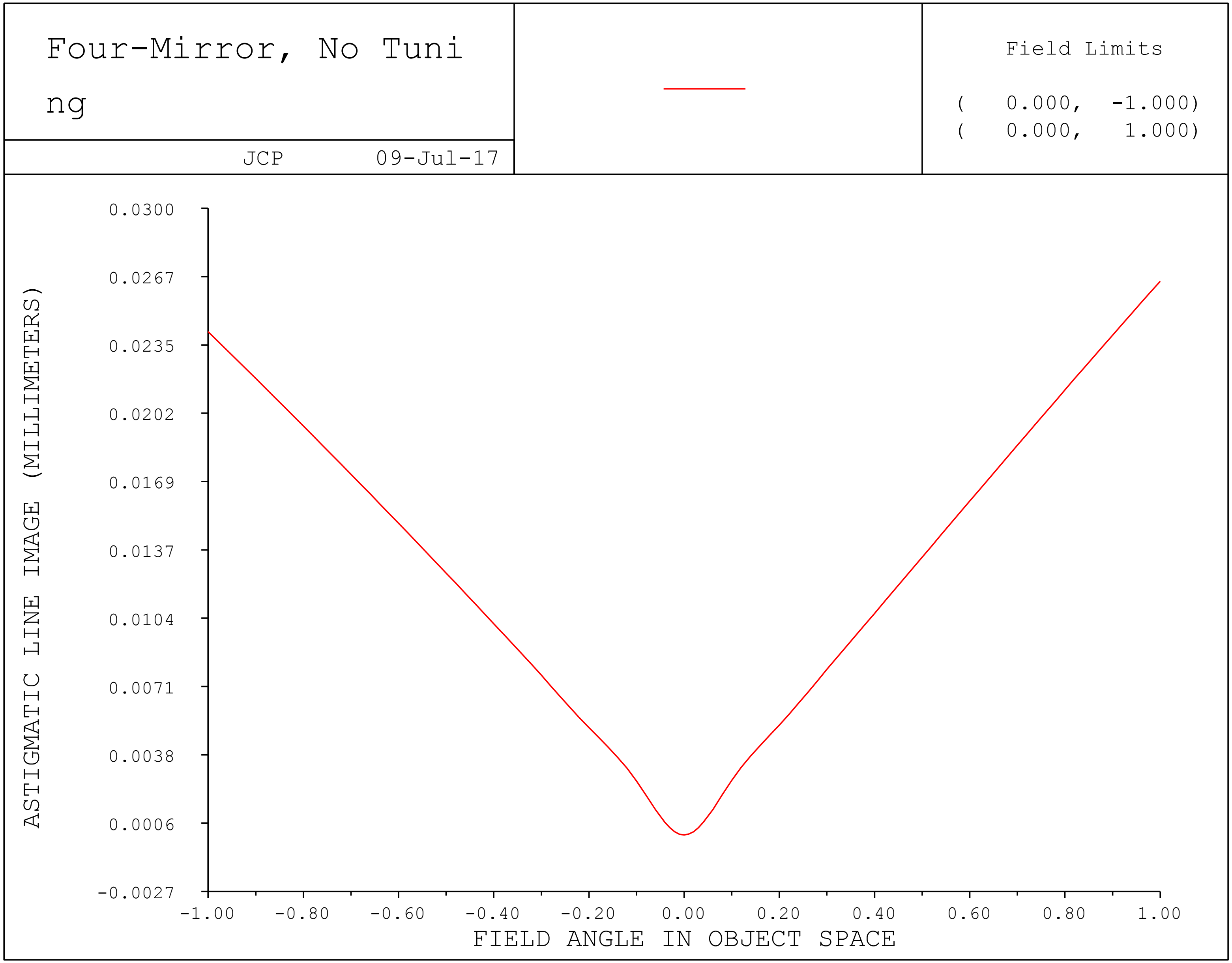
"A" Coefficients from On-Axis Equivalent from Before

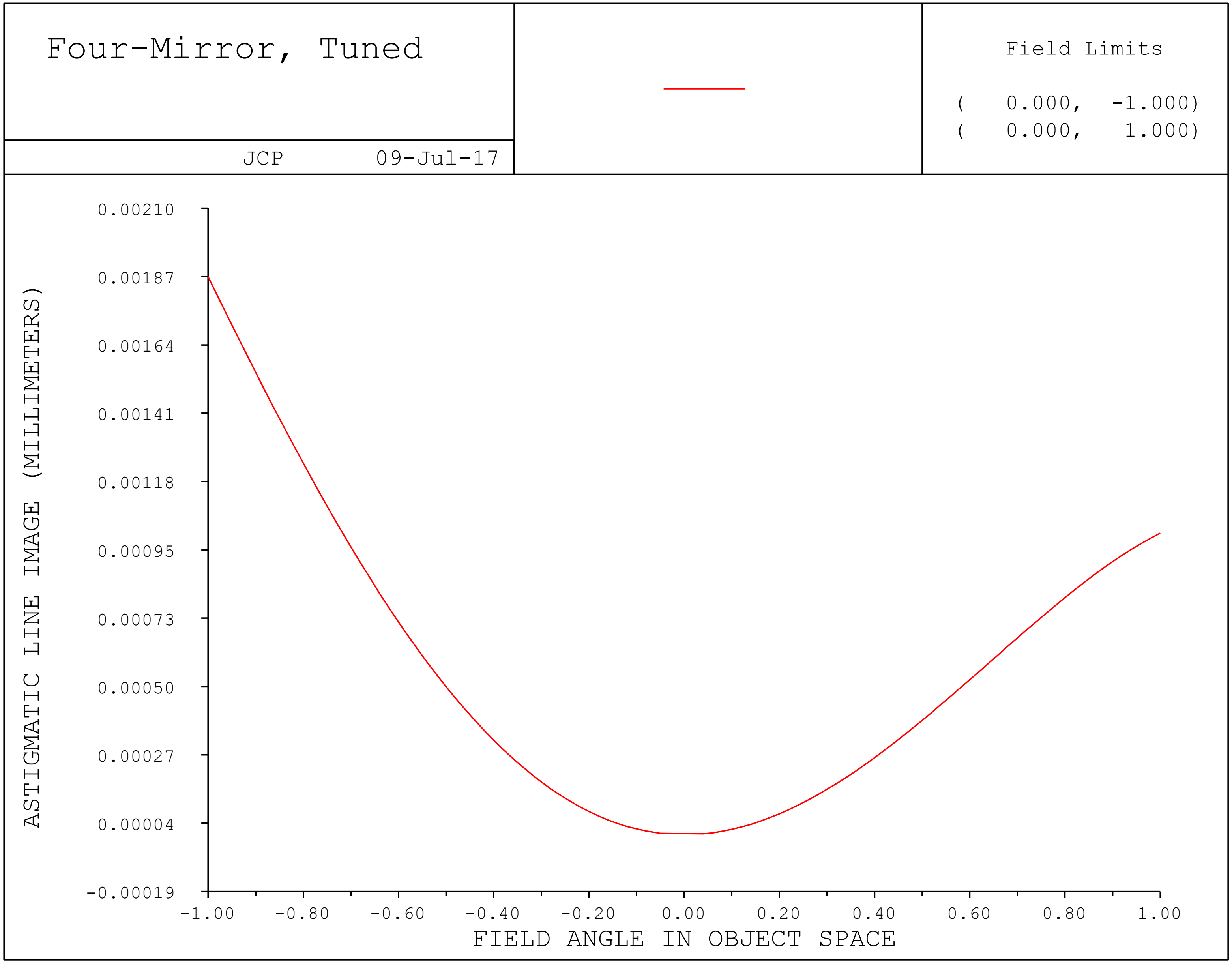
	M1			M2			M3			M4		
	Asphere	6.46E-12		Asphere	-5.05E-09		Asphere	-7.06E-10		Asphere	-6.38E-09	
	Existing	Needed	Sum	Existing	Needed	Sum	Existing	Needed	Sum	Existing	Needed	Sum
x^4	2.94E-10	6.46E-12	3.00E-10	8.10E-10	-5.05E-09	-4.24E-09	-2.93E-11	-7.06E-10	-7.35E-10	2.66E-09	-6.38E-09	-3.72E-09
x^3*y	0.00E+00	0	0.00E+00	0.00E+00	0	0.00E+00	0.00E+00	0	0.00E+00	0.00E+00	0	0.00E+00
x^2*y^2	5.47E-10	1.29E-11	5.60E-10	2.97E-09	-1.01E-08	-7.13E-09	7.78E-12	-1.41E-09	-1.40E-09	4.99E-09	-1.28E-08	-7.77E-09
x*y^3	0.00E+00	0	0.00E+00	0.00E+00	0	0.00E+00	0.00E+00	0	0.00E+00	0.00E+00	0	0.00E+00
y^4	2.55E-10	6.46E-12	2.61E-10	2.14E-09	-5.05E-09	-2.91E-09	3.63E-11	-7.06E-10	-6.70E-10	2.34E-09	-6.38E-09	-4.04E-09

-0.1375 waves   -2.0949 waves   3.1585 waves   -0.9260 waves

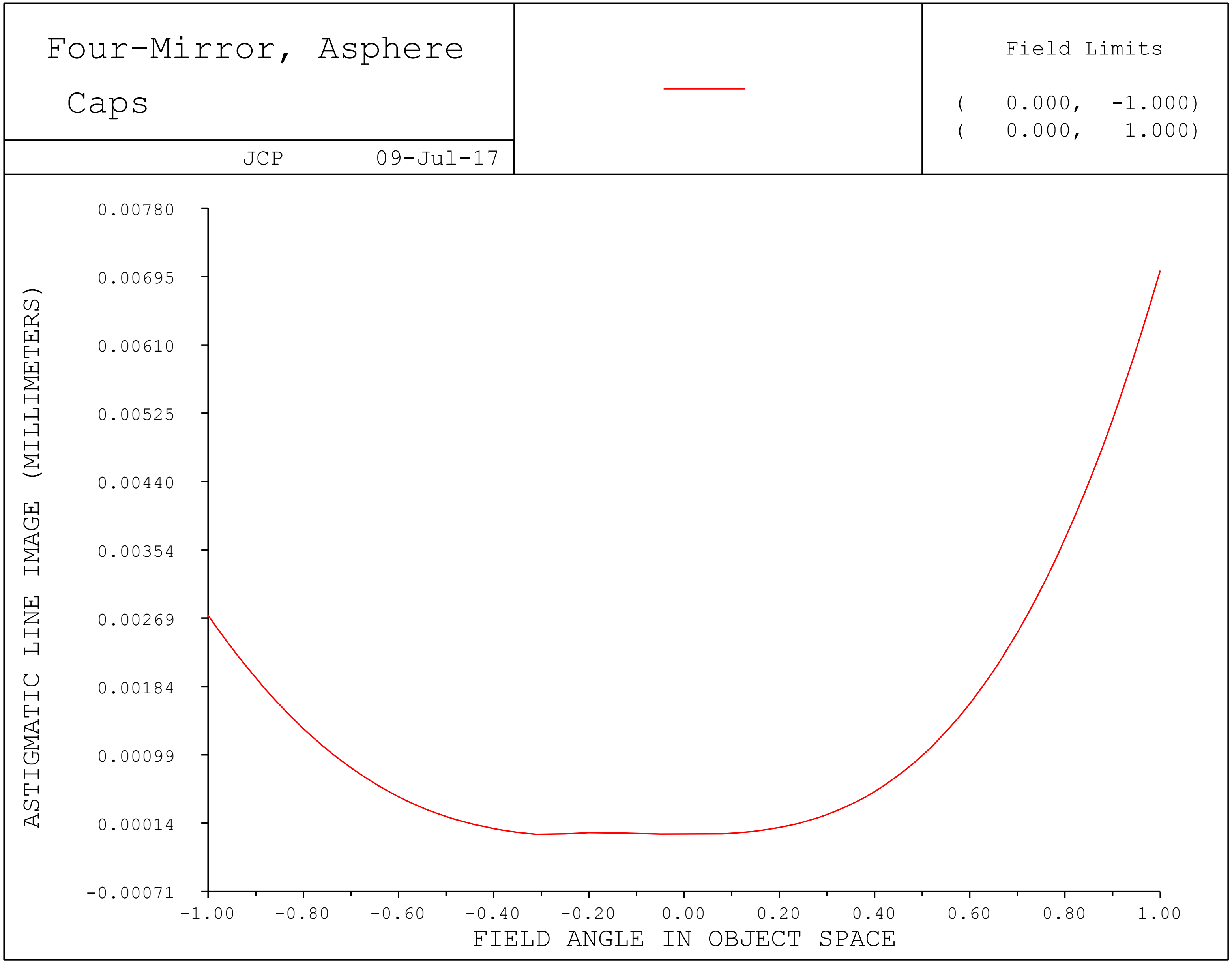
Spherical Aberration Contribution of Caps


On Axis Version
JCP 09-Jul-17
Field Limits
(0.000, -1.000)
(0.000, 1.000)
ASTIGMATIC LINE IMAGE (MILLIMETERS)
0.99E-03
0.88E-03
0.77E-03
0.67E-03
0.56E-03
0.45E-03
0.34E-03
0.23E-03
0.13E-03
0.18E-04
-.90E-04
-1.00 -0.80 -0.60 -0.40 -0.20 0.00 0.20 0.40 0.60 0.80 1.00
FIELD ANGLE IN OBJECT SPACE


Four-Mirror, No Tuning
JCP 09-Jul-17
Field Limits
(0.000, -1.000)
(0.000, 1.000)
0.0300
0.0267
0.0235
0.0202
0.0169
0.0137
0.0104
0.0071
0.0038
0.0006
-0.0027
ASTIGMATIC LINE IMAGE (MILLIMETERS)
-1.00
-0.80
-0.60
-0.40
-0.20
0.00
0.20
0.40
0.60
0.80
1.00
FIELD ANGLE IN OBJECT SPACE



Four-Mirror, Tuned
JCP 09-Jul-17
Field Limits
(0.000, -1.000)
(0.000, 1.000)
0.00210
0.00187
0.00164
0.00141
0.00118
0.00095
0.00073
0.00050
0.00027
0.00004
-0.00019
ASTIGMATIC LINE IMAGE (MILLIMETERS)
-1.00
-0.80
-0.60
-0.40
-0.20
0.00
0.20
0.40
0.60
0.80
1.00
FIELD ANGLE IN OBJECT SPACE


Four-Mirror, Asphere
Caps
JCP 09-Jul-17
Field Limits
(0.000, -1.000)
(0.000, 1.000)
ASTIGMATIC LINE IMAGE (MILLIMETERS)
0.00780
0.00695
0.00610
0.00525
0.00440
0.00354
0.00269
0.00184
0.00099
0.00014
-0.00071
-1.00
-0.80
-0.60
-0.40
-0.20
0.00
0.20
0.40
0.60
0.80
1.00
FIELD ANGLE IN OBJECT SPACE

Aberration Coefficients (Spherical Contribution)

$$A = ni = n\left(u + \frac{x\cos(I)}{R}\right) \text{ and } \bar{A} = n\bar{i} = n\left(\bar{u} + \frac{\bar{x}\cos(I)}{R}\right)$$

$$S_{\mathrm{I}} = -\frac{1}{8}A^2\Delta\left(\frac{u}{n}\right)x ,$$

$$S_{\mathrm{II}} = -\frac{1}{4R}\Psi^2\Delta\left(\frac{\cos(I)}{n}\right) ,$$

$$J_{\mathrm{I}} = -\frac{1}{2}n^2 \sin^2(I)\Delta\left(\frac{u}{n}\right)x ,$$

$$J_{\mathrm{II}} = -\frac{1}{2}n \sin(I)A\Delta\left(\frac{u}{n}\right)x ,$$

$$J_{\mathrm{III}} = -n \sin(I)\Psi\Delta\left(\frac{u}{n}\right) ,$$

$$J_{\mathrm{IV}} = -\frac{1}{2}\frac{n \sin(I)}{R}\Psi\Delta\left(\frac{1}{n}\right)x ,$$

$$J_{\mathrm{V}} = -\frac{1}{2}n \sin(I)\Psi^2\Delta\left(\frac{1}{n^2}\right)\frac{1}{x} ,$$

$$W_{02002} = \sum_{i=1}^{i=j}\{J_{\mathrm{I}}\}_i \quad \text{Constant astigmatism} ,$$

$$W_{11011} = \sum_{i=1}^{i=j}\left\{2\frac{\bar{x}}{x}J_{\mathrm{I}}\right\}_i \quad \text{Anamorphism} ,$$

$$W_{20020} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{x}}{x}\right)^2 J_{\mathrm{I}}\right\}_i \quad \text{Quadratic piston} ,$$

$$W_{03001} = \sum_{i=1}^{i=j}\{J_{\mathrm{II}}\}_i \quad \text{Constant coma} ,$$

$$W_{12101} = \sum_{i=1}^{i=j}\left\{2\frac{\bar{x}}{x}J_{\mathrm{II}} + J_{\mathrm{III}}\right\}_i \quad \text{Linear astigmatism} ,$$

$$W_{21001} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{x}}{x}\right)^2 J_{\mathrm{II}} + \frac{\bar{x}}{x}J_{\mathrm{III}} + J_{\mathrm{V}}\right\}_i$$

Quadratic distortion I ,

$$W_{12010} = \sum_{i=1}^{i=j}\left\{\frac{\bar{x}}{x}J_{\mathrm{II}} + J_{\mathrm{IV}}\right\}_i \quad \text{Field tilt} ,$$

$$W_{21110} = \sum_{i=1}^{i=j}\left\{2\left(\frac{\bar{x}}{x}\right)^2 J_{\mathrm{II}} + \frac{\bar{x}}{x}(J_{\mathrm{III}} + 2J_{\mathrm{IV}})\right\}_i$$

Quadratic distortion II ,

$$W_{30010} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{x}}{x}\right)^3 J_{\mathrm{II}} + \left(\frac{\bar{x}}{x}\right)^2 (J_{\mathrm{III}} + J_{\mathrm{IV}}) + \frac{\bar{x}}{x}J_{\mathrm{V}}\right\}_i$$

Cubic piston ,

$$W_{04000} = \sum_{i=1}^{i=j}\{S_{\mathrm{I}}\}_i \quad \text{Spherical aberration} ,$$

$$W_{13100} = \sum_{i=1}^{i=j}\left\{4\left(\frac{\bar{A}}{A}\right)S_{\mathrm{I}}\right\}_i \quad \text{Linear coma} ,$$

$$W_{22200} = \sum_{i=1}^{i=j}\left\{4\left(\frac{\bar{A}}{A}\right)^2 S_{\mathrm{I}}\right\}_i \quad \text{Quadratic astigmatism} ,$$

$$W_{22000} = \sum_{i=1}^{i=j}\left\{2\left(\frac{\bar{A}}{A}\right)^2 S_{\mathrm{I}} + S_{\mathrm{II}}\right\}_i \quad \text{Field curvature} ,$$

$$W_{31100} = \sum_{i=1}^{i=j}\left\{4\left(\frac{\bar{A}}{A}\right)^3 S_{\mathrm{I}} + 2\left(\frac{\bar{A}}{A}\right)S_{\mathrm{II}}\right\}_i \quad \text{Cubic distortion}$$

$$W_{40000} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{A}}{A}\right)^4 S_{\mathrm{I}} + \left(\frac{\bar{A}}{A}\right)^2 S_{\mathrm{II}}\right\}_i \quad \text{Quartic piston} .$$

Non-Spherical Contributions

$$Z_\alpha = \alpha\Delta[n\ \cos(I)]x^2\ ,$$
$$Z_\beta = \beta\Delta[n\ \cos(I)]x^3\ ,$$
$$Z_\gamma = \gamma\Delta[n\ \cos(I)]x^4\ .$$

Choose these, so that these cancel the field constant contributions

$$\Delta W_{02002} = Z_\alpha \quad \text{Constant astigmatism ,}$$
$$\Delta W_{11011} = 2\left(\frac{\bar{x}}{x}\right)Z_\alpha \quad \text{Anamorphism ,}$$
$$\Delta W_{20020} = \left(\frac{\bar{x}}{x}\right)^2 Z_\alpha \quad \text{Quadratic piston ,}$$
$$\Delta W_{03001} = Z_\beta \quad \text{Constant coma ,}$$
$$\Delta W_{12101} = 2\frac{\bar{x}}{x}Z_\beta \quad \text{Linear astigmatism ,}$$
$$\Delta W_{21001} = \left(\frac{\bar{x}}{x}\right)^2 Z_\beta \quad \text{Quadratic distortion I ,}$$
$$\Delta W_{12010} = \frac{\bar{x}}{x}Z_\beta \quad \text{Field tilt ,}$$
$$\Delta W_{21110} = 2\left(\frac{\bar{x}}{x}\right)^2 Z_\beta \quad \text{Quadratic distortion II ,}$$
$$\Delta W_{30010} = \left(\frac{\bar{x}}{x}\right)^3 Z_\beta \quad \text{Cubic piston ,}$$

$$\Delta W_{04000} = Z_\gamma \quad \text{Spherical aberration ,}$$
$$\Delta W_{13100} = 4\frac{\bar{x}}{x}Z_\gamma \quad \text{Linear coma ,}$$
$$\Delta W_{22200} = 4\left(\frac{\bar{x}}{x}\right)^2 Z_\gamma \quad \text{Quadratic astigmatism ,}$$
$$\Delta W_{22000} = 2\left(\frac{\bar{x}}{x}\right)^2 Z_\gamma \quad \text{Field curvature ,}$$
$$\Delta W_{31100} = 4\left(\frac{\bar{x}}{x}\right)^3 Z_\gamma \quad \text{Cubic distortion ,}$$
$$\Delta W_{40000} = \left(\frac{\bar{x}}{x}\right)^4 Z_\gamma \quad \text{Quartic piston .}$$

Remaining Coefficients after Adding Non-Spherical Contributions from Off-Axis Conics

$$A = ni = n\left(u + \frac{x\cos(I)}{R}\right) \text{ and } \bar{A} = n\bar{i} = n\left(\bar{u} + \frac{\bar{x}\cos(I)}{R}\right)$$

$$S_{\mathrm{I}} = -\frac{1}{8}A^2\Delta\left(\frac{u}{n}\right)x \, ,$$

$$S_{\mathrm{II}} = -\frac{1}{4R}\Psi^2\Delta\left(\frac{\cos(I)}{n}\right) \, ,$$

$$J_{\mathrm{I}} = -\frac{1}{2}n^2 \sin^2(I)\Delta\left(\frac{u}{n}\right)x \, ,$$

$$J_{\mathrm{II}} = -\frac{1}{2}n \sin(I)A\Delta\left(\frac{u}{n}\right)x \, ,$$

$$J_{\mathrm{III}} = -n \sin(I)\Psi\Delta\left(\frac{u}{n}\right) \, ,$$

$$J_{\mathrm{IV}} = -\frac{1}{2}\frac{n \sin(I)}{R}\Psi\Delta\left(\frac{1}{n}\right)x \, ,$$

$$J_{\mathrm{V}} = -\frac{1}{2}n \sin(I)\Psi^2\Delta\left(\frac{1}{n^2}\right)\frac{1}{x} \, ,$$

~~$W_{02002} = \sum_{i=1}^{i=j}\{J_{\mathrm{I}}\}_i$ Constant astigmatism ,~~

~~$W_{11011} = \sum_{i=1}^{i=j}\left\{2\frac{\bar{x}}{x}J_{\mathrm{I}}\right\}_i$ Anamorphism ,~~

Anamorphism conveniently cancels too

$$W_{20020} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{x}}{x}\right)^2 J_{\mathrm{I}}\right\}_i \quad \text{Quadratic piston ,}$$

~~$W_{03001} = \sum_{i=1}^{i=j}\{J_{\mathrm{II}}\}_i$ Constant coma ,~~

$$W_{12101} = \sum_{i=1}^{i=j}\left\{2\frac{\bar{x}}{x}J_{\mathrm{II}} + J_{\mathrm{III}}\right\}_i \quad \text{Linear astigmatism}$$

$$W_{21001} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{x}}{x}\right)^2 J_{\mathrm{II}} + \frac{\bar{x}}{x}J_{\mathrm{III}} + J_{\mathrm{V}}\right\}_i$$

Quadratic distortion I ,

$$W_{12010} = \sum_{i=1}^{i=j}\left\{\frac{\bar{x}}{x}J_{\mathrm{II}} + J_{\mathrm{IV}}\right\}_i \quad \text{Field tilt ,}$$

$$W_{21110} = \sum_{i=1}^{i=j}\left\{2\left(\frac{\bar{x}}{x}\right)^2 J_{\mathrm{II}} + \frac{\bar{x}}{x}(J_{\mathrm{III}} + 2J_{\mathrm{IV}})\right\}_i$$

Quadratic distortion II ,

$$W_{30010} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{x}}{x}\right)^3 J_{\mathrm{II}} + \left(\frac{\bar{x}}{x}\right)^2 (J_{\mathrm{III}} + J_{\mathrm{IV}}) + \frac{\bar{x}}{x}J_{\mathrm{V}}\right\}_i$$

Cubic piston ,

~~$W_{04000} = \sum_{i=1}^{i=j}\{S_{\mathrm{I}}\}_i$ Spherical aberration ,~~

$$W_{13100} = \sum_{i=1}^{i=j}\left\{4\left(\frac{\bar{A}}{A}\right)S_{\mathrm{I}}\right\}_i \quad \text{Linear coma ,}$$

$$W_{22200} = \sum_{i=1}^{i=j}\left\{4\left(\frac{\bar{A}}{A}\right)^2 S_{\mathrm{I}}\right\}_i \quad \text{Quadratic astigmatism ,}$$

$$W_{22000} = \sum_{i=1}^{i=j}\left\{2\left(\frac{\bar{A}}{A}\right)^2 S_{\mathrm{I}} + S_{\mathrm{II}}\right\}_i \quad \text{Field curvature ,}$$

$$W_{31100} = \sum_{i=1}^{i=j}\left\{4\left(\frac{\bar{A}}{A}\right)^3 S_{\mathrm{I}} + 2\left(\frac{\bar{A}}{A}\right)S_{\mathrm{II}}\right\}_i \quad \text{Cubic distortion}$$

$$W_{40000} = \sum_{i=1}^{i=j}\left\{\left(\frac{\bar{A}}{A}\right)^4 S_{\mathrm{I}} + \left(\frac{\bar{A}}{A}\right)^2 S_{\mathrm{II}}\right\}_i \quad \text{Quartic piston .}$$

Screenshot of Geogebra Tool