Exploring Space Through MATH

Applications in Precalculus

EDUCATOR EDITION

Space Vectors

Instructional Objectives

The 5-E's Instructional Model (Engage, Explore, Explain, Extend, and Evaluate) will be used to accomplish the following objectives.

Students will

- transform spherical coordinates to Cartesian coordinates;
- represent 3D vectors in terms of $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \text{and} \, \vec{\mathbf{k}}$; and
- perform vector addition operations.

Prerequisites

Students should have prior knowledge of spherical coordinates, azimuth, elevation, range, and vector notation. If students have no prior knowledge of spherical coordinates, teachers should introduce the spherical coordinate system. Teachers may use a three-dimensional model, on which the distance and two of the angles may be defined.

Background

This problem applies mathematical principles in NASA's human spaceflight.

The Western Aeronautical Test Range (WATR), located at NASA Dryden Flight Research Center in Edwards, California, provides range engineering and technical expertise and resources to support aerospace research, science, and low-Earth orbiting missions. High-accuracy radar provides tracking and space positioning information on the International Space Station (ISS), as it has previously done with other research vehicles, such as the space shuttle.

During space shuttle missions, the WATR Aeronautical Tracking Facility (ATF) provided telemetry (information collected via radio waves), radar, voice communication, and video support of ISS and space shuttle activities to NASA Johnson Space Center in Houston, Texas using various telemetry tracking, space positioning, and audio communication systems.

The telemetry tracking system provided status information on the condition of the space shuttle (and the pilot's point of view video when available) to the NASA network via satellite. When required, the telemetry systems also provided uplinked command data to the space shuttle.

Key Concepts
Vector addition

Problem Duration
60 minutes

Technology
Calculator, projector with movie player, computer with internet access, Google Earth

Materials
- Student Edition: *Space Vectors*
- Globe or spherical object
- Presentation: *Spherical Coordinate Systems.pps*

Skills
Coordinate system transformations, vector representation, vector addition

NCTM Standards
- Numbers and Operations
- Algebra
- Geometry

Common Core Standards
- Algebra
- Number and Quantity Overview

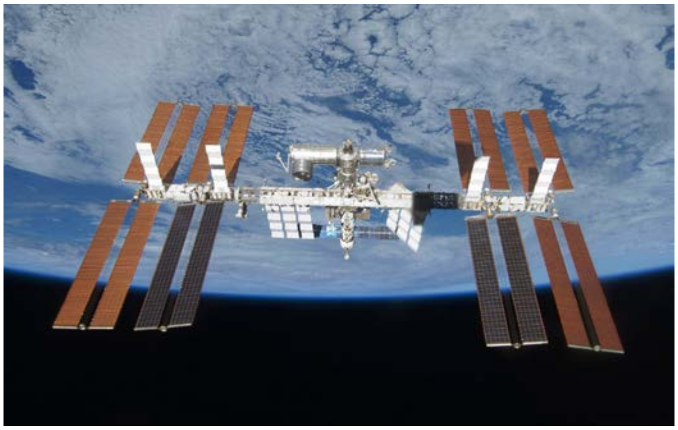

Figure 1: The International Space Station in low-Earth orbit

Figure 2: WATR instrumentation radar

The space positioning system consists of two high-accuracy radars, differential global positioning system ground stations, and Federal Aviation Administration surveillance radar data. This system was used to track every space shuttle orbit, relaying time-space positioning information from launch to landing. The space positioning system also tracked the ISS—from the day prior to the space shuttle launch and through the duration of the mission—providing critical information for the docking and undocking of the space shuttle.

Throughout each space shuttle mission, voice communication was enabled by the audio communication system. While a system of communication satellites used by NASA and other United States government agencies (known as the Tracking and Data Relay Satellite System, or TDRSS) provided the primary voice communication for the space shuttle, the WATR provided back-up support in case of a communications failure during a mission. The WATR also became the primary means of communication support in the event a space shuttle might be diverted locally to Edwards Air Force Base for a landing.

Mission data from these three systems were processed near real-time, and were archived by NASA as a means of support for post-mission analyses.

NCTM Principles and Standards

Numbers and Operations

- Develop an understanding of properties of, and representations for, the addition and multiplication vectors and matrices
- Develop fluency in operation with real numbers, vectors, matrices, using mental computation or pencil-and-paper for simple cases and technology for more complicated cases
- Judge the reasonableness of numerical computations and their results

Algebra

- Use symbolic algebra to represent and explain mathematical relationships

Geometry

- Use Cartesian coordinates and other coordinate systems, such as navigational, polar, or spherical, to analyze geometric situations

Common Core Standards

Algebra

- Create equations that describe numbers or relationships

Number and Quantity Overview

- Vector and matrix quantities

Lesson Development

Following are the phases of the 5-E's model in which students can construct new learning based on prior knowledge and experiences. The time allotted for each activity is approximate. Depending on class length, the lesson may be broken into multiple class periods.

1 – Engage (15 minutes)

- Download the *Spherical Coordinate Systems* PowerPoint, and open the file. This animation will give students an introduction to reading maps and understanding three-dimensional coordinates. It will also help students understand the navigation terms in this activity. *Note: Several of the terms used in the Spherical Coordinate Systems animation maybe helpful when students reach question 3. Consider using it then.*
- With students working in pairs, have them read the background section and come up with one or two brief summary statements from their reading. Allow students to share their summary statements with the class.

2 – Explore (15 minutes)

- Download and open Google Earth using a projector so students may observe. http://www.google.com/earth/download/ge/agree.html
- Distribute the student worksheet, *Space Vectors.*
- Have students work as a class to answer questions 1–2.

3 – Explain (15 minutes)

- Have students work in pairs to answer questions 3–5.
- Call on students to give their answers and discuss.

4 – Extend (5 minutes)

- Have students work in pairs to answer question 6.
- Encourage student discussion, and ask if there are any questions.

5 – Evaluate (10 minutes)

- Have students work independently to complete question 7.
- This may be done in class or assigned as homework.

Space Vectors

Solution Key

Problem

During a space shuttle flight, the Flight Dynamics Officer (FDO) monitors the location and performance of the space shuttle, both in atmosphere and in orbit. This flight controller is in charge of the location and destination of the space shuttle. The FDO calculates orbital maneuvers and resulting trajectories using position and velocity vectors. Radar positioned at NASA Dryden Flight Research Center (FRC) in Edwards, California collects position data of the space shuttle.

Directions: Show all work and justify your answers. Answer questions 1–2 with your class and questions 3–4 with your partner. Round all answers to the nearest thousandth and label with the appropriate units.

1. Use Google Earth to find the location of the coordinates in Table 1. Enter your coordinates in terms of latitude and longitude. Remember that Google Earth divides the Earth into hemispheres with each being only 180 degrees. (For example: *40° S, 56° E*)

   The location is near Edwards Air Force Base in California. Students must recognize that they must subtract the longitude from 360 degrees to get standard spherical coordinates of 34.9607796° N, 117.911496° W. Students will need to zoom out to see that it is near Edwards Air Force Base and in California. If students zoom in, they will be able to see the outline of the radar that is used for tracking.

2. Use Google Earth to find your school. Write the coordinates below using NASA's naming conventions for longitude and latitude (which does not divide the Earth into hemispheres).

   Answers will vary. For example, J. Frank Dobie High School in Houston, Texas is located at 29.38° N, 95.15° W. When these coordinates are placed in the NASA naming convention, they would read 29.38° latitude, 264.85° longitude.

Table 1 shows the position of the FRC radar. NASA's latitudinal radar readings are expressed in degrees north of the equator, while longitudinal radar readings are expressed in degrees east of the prime meridian. The FRC radius is the distance from the center of the Earth to the FRC itself.

Table 1: April 2010 STS-131 Radar Position

FRC Latitude (ϕ) deg	FRC Longitude (λ) deg	FRC Radius (r) km
34.9607796	242.0885039	6378.889

In order to complete the vector analysis, azimuth, elevation, and range are also required. Azimuth is the horizontal angle of the radar measured clockwise from a line pointing due north. Elevation is the angle of the radar above the local horizon. Range is the distance along the geometric line-of-sight from the radar to the vehicle. Figure 3 is an illustration of these orbital terms.

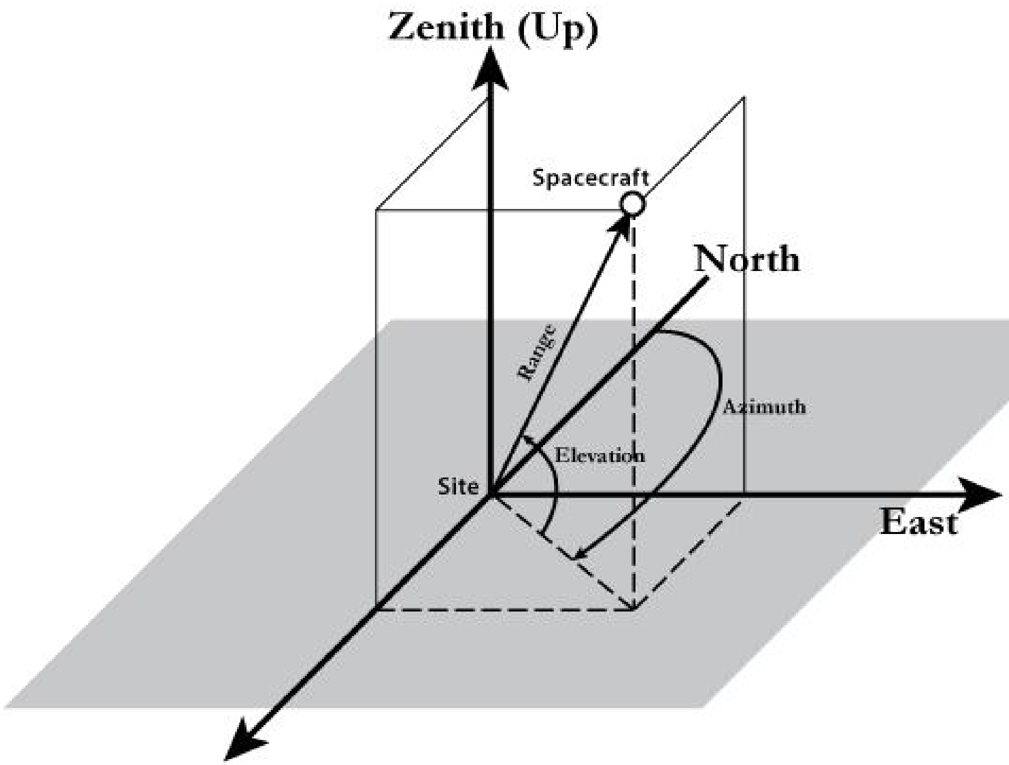

Figure 3: Illustration of orbital terms

3. The FRC data found in Table 1 was used to determine the vector that represented the position of the FRC site with respect to the Earth's center in terms of $\vec{\mathbf{i}}, \vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$.

Conversion equations (spherical to Cartesian coordinates)

$$x = r\cos\phi\cos\lambda$$
$$y = r\cos\phi\sin\lambda$$
$$z = r\sin\phi$$

r = FRC radius, ϕ = latitude, and λ = longitude

a. Find the *x*-coordinate of the FRC.

$$x = r\cos\phi\cos\lambda$$
$$x = 6378.889\cos(34.9607796^\circ)\cos(242.0885039^\circ)$$
$$x = -2447.163\ km$$

b. Find the *y*-coordinate of the FRC.

$$y = r\cos\phi\sin\lambda$$
$$y = 6378.889\cos(34.9607796^\circ)\sin(242.0885039^\circ)$$
$$y = -4619.644\ km$$

c. Find the z-coordinate where ϕ = latitude.

$$z = r \sin \phi$$
$$z = 6378.889 \sin(34.9607796°)$$
$$z = 3655.203 \text{ km}$$

d. Write the Earth-centered vector, $\vec{\mathbf{f}}_{ec}$, in the form: $\vec{\mathbf{f}}_{ec} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$.

$$\vec{f}_{ec} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$\vec{f}_{ec} = -2447.163\vec{i} - 4619.644\vec{j} + 3655.203\vec{k}$$

4. The FRC also collects data that can be used to determine the vector that represents the position of the space shuttle with respect to the FRC site in terms of **i**, **j**, and **k**, where *El* = elevation of the spacecraft, *Az* = azimuth, and ρ = range. Table 2 below shows the spherical coordinates for the space shuttle's position.

Table 2: Position of the space shuttle with respect to the FRC

STS-131 FRC Elevation angle (*El*) deg	**STS-131 FRC Azimuth angle (*Az*) deg**	**STS-131 FRC Range to target (ρ) km**
40.8300297	199.9850926	505.6889904

Note: Conversion equations (topodetic to Cartesian coordinates) relative to site located at (ϕ,λ) Topodetic refers to a non-rotating frame of reference where the origin is the site location given by its longitude and latitude.

$$x = -\rho\cos(El)\cos(Az)\sin(\phi)\cos(\lambda) - \rho\cos(El)\sin(Az)\sin(\lambda) + \rho\sin(El)\cos(\phi)\cos(\lambda)$$
$$y = -\rho\cos(El)\cos(Az)\sin(\phi)\sin(\lambda) + \rho\cos(El)\sin(Az)\cos(\lambda) + \rho\sin(El)\cos(\phi)\sin(\lambda)$$
$$z = \rho\cos(El)\cos(Az)\cos(\phi) + \rho\sin(El)\sin(\phi)$$

El = elevation angle, *Az* = azimuth angle, ρ = range, ϕ = latitude, and λ = longitude

a. Find the x-coordinate of the space shuttle's position relative to FRC.

$$x = -\rho\cos(El)\cos(Az)\sin(\phi)\cos(\lambda) - \rho\cos(El)\sin(Az)\sin(\lambda) + \rho\sin(El)\cos(\phi)\cos(\lambda)$$
$$x = -505.6889904\cos(40.8300297°)\cos(199.9850926°)\sin(34.9607796°)\cos(242.0885039°)$$
$$-505.6889904\cos(40.8300297°)\sin(199.9850926°)\sin(242.0885039°)$$
$$+505.6889904\sin(40.8300297°)\cos(34.9607796°)\cos(242.0885039°)$$
$$x = -338.855 \text{ km}$$

b. Find the *y*-coordinate of the space shuttle's position relative to FRC.

$$y = -\rho\, cos(El)\, cos(Az)\, sin(\phi)\, sin(\lambda) + \rho\, cos(El)\, sin(Az)\, cos(\lambda) + \rho\, sin(El)\, cos(\phi)\, sin(\lambda)$$
$$y = -505.6889904\, cos(40.8300297°)\, cos(199.9850926°)\, sin(34.9607796°)\, sin(242.0885039°)$$
$$+\rho\, cos(40.8300297°)\, sin(199.9850926°)\, cos(242.0885039°)$$
$$+\rho\, sin(40.8300297°)\, cos(34.9607796°)\, sin(242.0885039°)$$
$$y = -360.309 \text{ km}$$

c. Find the *z*-coordinate of the space shuttle's position relative to FRC.

$$z = \rho\, cos(El)\, cos(Az)\, cos(\phi) + \rho\, sin(El)\, sin(\phi)$$
$$z = 505.6889904\, cos(40.8300297°)\, cos(199.9850926°)\, cos(34.9607796°)$$
$$+505.6889904\, sin(40.8300297°)\, sin(34.9607796°)$$
$$z = -105.245 \text{ km}$$

d. Write the site-centered vector, $\vec{\mathbf{s}}_{fc}$, in the form: $\vec{\mathbf{s}}_{fc} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$.

$$\vec{s}_{fc} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$\vec{s}_{fc} = -338.855\vec{i} - 360.309\vec{j} - 105.245\vec{k}$$

Directions: Answer questions 5–6 with your partner. Discuss the answers to be sure that you both understand and agree on the solution. Round all answers to the nearest thousandth and label with the appropriate units.

5. With knowledge of the FRC site position and the space shuttle position (with respect to the FRC), the Flight Dynamics Officer can determine the space shuttle's position with respect to the Earth's center using vector addition. The resultant vector, $\vec{\mathbf{s}}_{ec}$, gives the position of the space shuttle with respect to the Earth's center. Find the Earth-centered position vector.

$$\vec{s}_{ec} = \vec{f}_{ec} + \vec{s}_{fc}$$
$$\vec{s}_{ec} = (-2447.163\vec{i} - 4619.644\vec{j} + 3655.203\vec{k}) + (-338.855\vec{i} - 360.309\vec{j} - 105.245\vec{k})$$
$$\vec{s}_{ec} = (-2447.163 - 338.855)\vec{i} + (-4619.644 - 360.309)\vec{j} + (3655.203 - 105.245)\vec{k}$$
$$\vec{s}_{ec} = -2786.018\vec{i} - 4979.953\vec{j} + 3549.958\vec{k}$$

6. The radius of Earth is 6378.137 km. Find the space shuttle's distance above Earth's surface.

The shuttle's distance is the difference between the magnitude of $\vec{\mathbf{s}}_{ec}$ *and the radius of Earth.*

$$h = \|\vec{s}_{ec}\| - r_{earth}$$
$$h = \sqrt{x^2 + y^2 + z^2} - r_{earth}$$
$$h = \sqrt{(-2786.018)^2 + (-4979.953)^2 + (3549.958)^2} - 6378.137$$
$$h = 342.282 \text{ km}$$

Directions: Complete question 7 independently. Round the answer to the nearest thousandth and label with the appropriate units.

7. Using the FRC position vector (determined in Question 3 Part d), determine the Earth-centered position vector, $\vec{\mathbf{s}}_{ec}$, of the space shuttle if the radar site had reported an elevation of 30° and an azimuth of 150°. Then give the Earth-centered coordinates of the shuttle.

   Find $\vec{\mathbf{s}}_{fc}$.

$$\vec{s}_{fc} = [-\rho\cos(El)\cos(Az)\sin(\phi)\cos(\lambda) - \rho\cos(El)\sin(Az)\sin(\lambda) + \rho\sin(El)\cos(\phi)\cos(\lambda)]\vec{\boldsymbol{i}}$$
$$+[-\rho\cos(El)\cos(Az)\sin(\phi)\sin(\lambda) + \rho\cos(El)\sin(Az)\cos(\lambda) + \rho\sin(El)\cos(\phi)\sin(\lambda)]\vec{\boldsymbol{j}}$$
$$+[\rho\cos(El)\cos(Az)\cos(\phi) + \rho\sin(El)\sin(\phi)]\vec{\boldsymbol{k}}$$
$$\vec{s}_{fc} = -5.234\vec{\boldsymbol{i}} - 477.658\vec{\boldsymbol{j}} - 165.942\vec{\boldsymbol{k}}$$

   Find $\vec{\boldsymbol{s}}_{ec}$.

   Find $\vec{\boldsymbol{s}}_{ec}$ by substituting in the following values: El = 30º, Az = 150º, ρ = 505.6889904*, ϕ = 34.9607796, and λ = 242.088904.*

$$\vec{s}_{ec} = \vec{\boldsymbol{f}}_{ec} + \vec{s}_{fc}$$
$$\vec{s}_{ec} = (-2447.163\vec{\boldsymbol{i}} - 4619.644\vec{\boldsymbol{j}} + 3655.203\vec{\boldsymbol{k}}) + (-5.234\vec{\boldsymbol{i}} - 477.658\vec{\boldsymbol{j}} - 165.942\vec{\boldsymbol{k}})$$
$$\vec{s}_{ec} = (-2447.163 - 5.234)\vec{\boldsymbol{i}} + (4619.644 - 477.658)\vec{\boldsymbol{j}} + (3655.203 - 165.942)\vec{\boldsymbol{k}}$$
$$\vec{s}_{ec} = -2452.397\vec{\boldsymbol{i}} - 5097.302\vec{\boldsymbol{j}} + 3489.261\vec{\boldsymbol{k}}$$

   The Earth-centered coordinates of the space shuttle are (-2452.397, -5097.302, 3489.261).

Contributors

This problem was developed by the Human Research Program Education and Outreach (HRPEO) team with the help of NASA subject matter experts and high school mathematics educators.

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