MATH AND SCIENCE @ WORK

AP* CALCULUS Educator Edition

LUNAR SURFACE COMMUNICATIONS

Instructional Objectives

Students will

- derive the formula for calculating the line of sight distance to a horizon tangent point;
- derive the distance along a surface to a tangent point; and
- use derivatives to find the rates of change of two or more variables that are changing with respect to distance.

Degree of Difficulty

This problem is challenging because students need to recall and apply mathematical concepts from Algebra I, Geometry, and Trigonometry.

- For the average AP Calculus AB/BC student the problem is moderately difficult.

Background

This problem is part of a series of problems that apply math and science to human space exploration at NASA.

Exploration provides the foundation of our knowledge, technology, resources, and inspiration. It seeks answers to fundamental questions about our existence, responds to recent discoveries and puts in place revolutionary techniques and capabilities to inspire our nation, the world, and the next generation. Through NASA, we touch the unknown, we learn and we understand. As we take our first steps toward sustaining a human presence in the solar system, we can look forward to far-off visions of the past becoming realities of the future.

Outpost concepts are now being designed and studied by engineers, scientists, and sociologists to facilitate long-duration human missions to the surface of the Moon or other planetary bodies (Figure 1). Such outposts will include habitat modules, laboratory modules, power systems, transportation, life support systems, and protection from the environment.

These long-duration missions will also require robust and reliable communications. It will be important to maintain constant communications with Earth. Therefore, 24 hours per day/7 days per week coverage at the outpost could be a requirement. This will likely be accomplished by a

Grade Level
11-12

Key Topic
Differentiation

Degree of Difficulty
Calculus AB: Moderate
Calculus BC: Moderate

Teacher Prep Time
15 minutes

Problem Duration
40-50 minutes

Technology
Graphing calculator

Materials
Student Edition including:
- Background handout
- Problem worksheet
- Support diagrams

AP Course Topics
Derivatives:
- Concept of the Derivative
- Derivative at a Point
- Applications of Derivatives
- Computation of Derivatives

NCTM Principles and Standards
- Algebra
- Geometry
- Problem Solving
- Connections

combination of communication satellites in orbit around the planetary body and communication equipment on the surface.

The habitat (Figure 1) on the surface will need video downlink capability to Earth. In addition to the communication requirements between the planetary surface and Earth, it will also be important to maintain constant communications between surface crew members, regardless of their distance from the outpost.

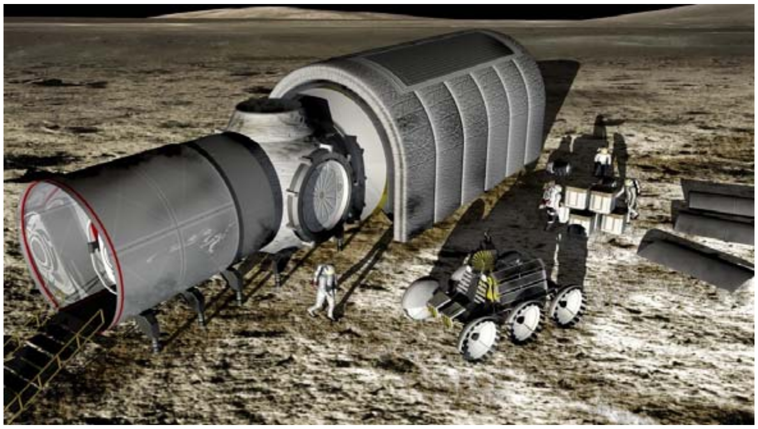


Figure 1: Habitat, airlock, and vehicles (NASA concept)

Surface to surface communications involves communicating between astronauts, rovers, robots, habitats, power stations, and science experiments, as well as communication within the habitats. For surface-based communication systems, there is a line of sight limitation on rover communication with the habitat. Astronauts must have either the habitat or the rover in their line of sight to maintain communications with Earth.

The communications system should be easily expandable. Future missions will not want to abandon existing equipment, but instead incorporate existing equipment into an expanding communications system.

These plans give NASA a huge head start in getting to Mars. We will already have rockets capable of transporting heavy cargo, as well as a versatile crew capsule. An outpost within a few days travel from Earth would give us needed practice of "living off the land" away from our home planet, before making the longer trek to Mars.

AP Course Topics

Derivatives

- Concept of the derivative
  - Derivative interpreted as an instantaneous rate of change
- Derivative at a Point
  - Tangent line to a curve at a point and local linear approximation
- Applications of derivatives
  - Modeling rates of change, including related rates problems
  - Interpretation of the derivative as a rate of change in varied applied contexts, including velocity, speed, and acceleration
- Computations of derivatives:
  - Chain rule and implicit differentiation

NCTM Principles and Standards

Algebra
- Understand patterns, relations, and functions.
- Represent and analyze mathematical situations and structures using algebraic symbols.
- Use mathematical models to represent and understand quantitative relationships.
- Analyze change in various contexts.

Geometry
- Use visualization, spatial reasoning, and geometric modeling to solve problems.

Problem Solving
- Build new mathematical knowledge through problem solving.
- Solve problems that arise in mathematics and in other contexts.

Connections
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.

Problem
When relying on surface to surface communication understanding line of sight is critical. Consequently, an important measurement in planetary exploration is the distance to the horizon. This depends on the diameter of the planet and the height of the observer above the surface. Geometry can determine the height of a transmission antenna required to insure proper reception within a specified distance. Use the diagram in Figure 2 to answer the following questions.

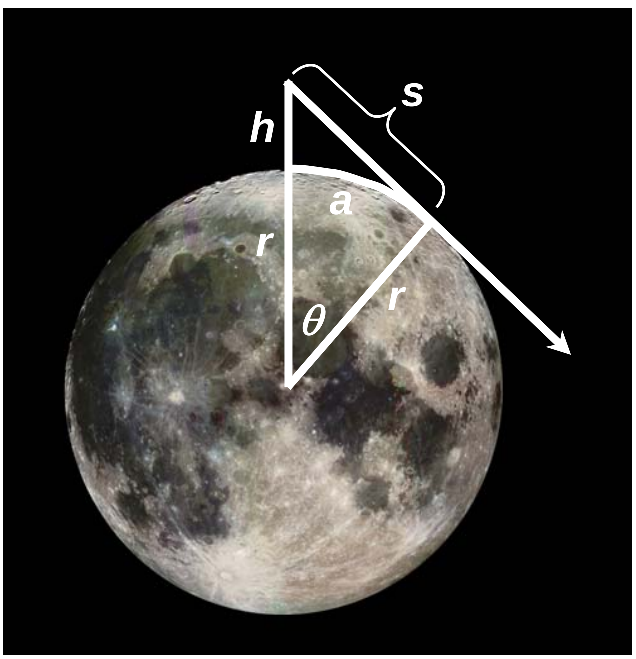


Figure 2: Problem Diagram of the Moon
NOTE: Diagram is exaggerated to show relationship and reference points.

A. If the radius of the Moon is given by *r*, and the height of the tower above the surface is given by *h*, use Figure 2 to derive the formula for the line of sight distance, *s*, to the horizon tangent point.

B. In terms of *r* and *h*, derive the formula for the arc length, *a*, which is the distance along the moon to the point of tangency.

C. On Earth, a radio station may have an antenna tower 50 meters (m) tall. What would be the reception distance *s*, to the nearest meter, if that same tower were on the Moon? The radius of the Moon is 1,738 kilometers (km).

D. Graph the equation for the line of sight distance from question A over the interval 0, 60. What happens as the antenna height increases? To the nearest meter, find the rate of change to the lunar line of sight with respect to the antenna height or *ds/dh* at *h* = 50 m. In practical terms, what does this mean?

E. Use local linear approximation to predict the distance for an antenna of 51 meters. How does this compare to the actual calculation using your equation from question A? Would local linear approximation be as accurate in predicting the distance for an antenna height of 11 meters? Explain your reasoning.

F. What is the rate of change of the distance, *a*, along the lunar surface to the lunar tower at the line of sight position when *h* = 50 m? Express your answer as a whole number.

G. Under what conditions do the line of sight formula (question D) and the arc length formula (question E) give significantly different answers? When would you use the arc length formula on the Moon or on some other solar system body?

Solution Key (One Approach)

A. If the radius of the Moon is given by *r*, and the height of the tower above the surface is given by *h*, use Figure 2 to derive the formula for the line of sight distance, *s*, to the horizon tangent point.

Use the Pythagorean Theorem to solve for s.

$$\begin{aligned} s^2 + r^2 &= (r+h)^2 \\ s^2 &= (r+h)^2 - r^2 \\ s^2 &= r^2 + 2rh + h^2 - r^2 \\ s^2 &= 2rh + h^2 \\ s &= \sqrt{2rh + h^2} \end{aligned}$$

B. In terms of *r* and *h*, derive the formula for the arc length, *a*, which is the distance along the moon to the point of tangency.

Start with the equation for arc length.

$$a = r\theta$$

Remember,

$$\cos\theta = \frac{r}{r+h}$$

so,

$$a = r\cos^{-1}\left(\frac{r}{r+h}\right)$$

C. On Earth, a radio station may have an antenna tower 50 meters (m) tall. What would be the reception distance *s*, to the nearest meter, if that same tower were on the Moon? The radius of the Moon is 1,738 kilometers (km).

Note: You might want to discuss why rounding to the nearest meter would be sufficient here rather than rounding to three decimal places because of the magnitude. Students might see that rounding to the nearest meter would be the same measure as rounding km to three decimal places.

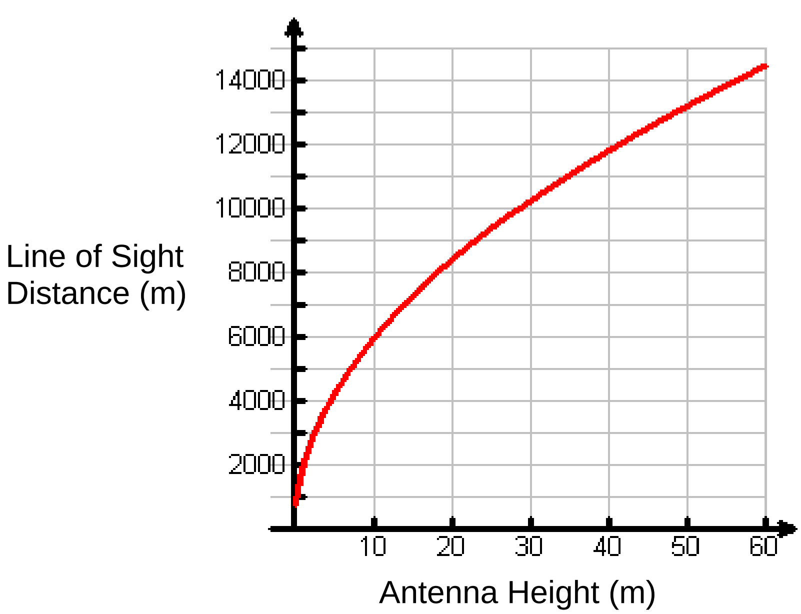
h = 50 m
r = 1,738 km or 1,738,000 m

$$s = \sqrt{2rh + h^2}$$
$$s = \sqrt{2 \cdot 1{,}738{,}000 \cdot 50 + 50^2}$$
$$s = 13{,}183\ m$$

Note: Because the radius of the Moon is significantly greater than the height of the tower, you may simplify the equation $s = \sqrt{2rh + h^2}$ *to* $s = \sqrt{2rh}$ *and arrive at the same answer.*

D. Graph the equation for the line of sight distance from question A over the interval 0, 60. What happens as the antenna height increases? To the nearest meter, find the rate of change to the lunar line of sight with respect to the antenna height or *ds/dh* at *h* = 50 m. In practical terms, what does this mean?

Line of Sight Distance vs. Antenna Height



The line of sight distance increases as the antenna height increases, but not at a constant rate. This rate decreases as the antenna height increases.

This particular example would be a good one for students to practice implicit differentiation to find the rate of change or the students could use the chain rule. Using implicit differentiation gives the following:

$$s^2 = 2rh + h^2$$

$$2s\frac{ds}{dh} = 2r + 2h$$

$$\frac{ds}{dh} = \frac{2r + 2h}{2s}$$

$$\frac{ds}{dh} = \frac{r + h}{s}$$

$$\frac{ds}{dh} = \frac{r + h}{\sqrt{2rh + h^2}}$$

When h = 50 meters,

$$\frac{ds}{dh} = \frac{1738000 + 50}{\sqrt{2(1738000)(50) + (50)^2}}$$

$$\frac{ds}{dh} = 132\,\frac{\text{m}}{\text{m}}$$

In practical terms, this means that the line of sight distance is increasing by 132 meters for every meter of antenna height.

E. Use local linear approximation to predict the distance for an antenna of 51 meters. How does this compare to the actual calculation using your equation from questionA? Would local linear approximation be as accurate in predicting the line of sight distance for an antenna height of 11 meters? Explain your reasoning.

Using local linear approximation we can then predict the line of sight distance for a 51 meter tower as follows:

$$f(51) = f(50) + f'(50)(51 - 50)$$

$$f(51) = 13183 + 132(1)$$

$$f(51) = 13315 \text{ meters}$$

Now we compare this to the distance found using the original equation:

$$f(51) = \sqrt{2(1738000)(51) + (51)^2}$$

$$f(51) = 13315 \text{ meters}$$

The local linear approximation is very accurate at h=51.

The concavity of the graph from question D shows that a prediction using local linear approximations would not be as accurate.

F. What is the rate of change of the distance, *a*, along the lunar surface to the lunar tower at the line of sight position when *h* = 50 m? Express your answer as a whole number.

Use the chain rule to find the rate of change.

$$\frac{da}{dh} = \frac{da}{du} \cdot \frac{du}{dh}$$

$$a = r \cos^{-1}\left(\frac{r}{r+h}\right)$$

Let $u = \dfrac{r}{r+h}$

then $a = r\cos^{-1}(u)$

$$\frac{da}{du} = r \cdot -1(1-u^2)^{-\frac{1}{2}} + \cos^{-1}(u) \cdot 0$$

$$\frac{da}{du} = -r(1-u^2)^{-\frac{1}{2}}$$

$$\frac{da}{du} = -r\left[1-\left(\frac{r}{r+h}\right)^2\right]^{-\frac{1}{2}}$$

$$\frac{du}{dh} = r \cdot -1(r+h)^{-2} + (r+h)^{-1} \cdot 0$$

$$\frac{du}{dh} = \frac{-r}{(r+h)^2}$$

$$\frac{da}{dh} = -r\left[1-\left(\frac{r}{r+h}\right)^2\right]^{-\frac{1}{2}} \cdot \frac{-r}{(r+h)^2}$$

$$\frac{da}{dh} = \frac{-r}{\left[1-\frac{r^2}{(r+h)^2}\right]^{\frac{1}{2}}} \cdot \frac{-r}{(r+h)^2}$$

$$\frac{da}{dh} = \frac{-r}{\left[\frac{(r+h)^2 - r^2}{(r+h)^2}\right]^{\frac{1}{2}}} \cdot \frac{-r}{(r+h)^2}$$

$$\frac{da}{dh} = \frac{r^2}{\left[(r+h)^2 - r^2\right]^{\frac{1}{2}} \cdot (r+h)}$$

$$\frac{da}{dh} = \frac{r^2}{(2rh + h^2)^{\frac{1}{2}} \cdot (r+h)}$$

$$\frac{da}{dh} = \frac{(1{,}738{,}000\ m)^2}{\left[2 \cdot 1{,}738{,}000\ m \cdot 50\ m + (50\ m)^2\right]^{\frac{1}{2}} \cdot (1{,}738{,}000\ m + 50\ m)}$$

$$\frac{da}{dh} = 131.8 \text{ meters per meter of tower height}$$

Note: Once again, because r is significantly greater than h, you may simplify the equation $\frac{da}{dh} = \frac{r^2}{(2rh + h^2)^{\frac{1}{2}} \cdot (r+h)}$ to: $\frac{da}{dh} = \frac{r}{(2rh)^{\frac{1}{2}}}$ or $\frac{da}{dh} = \frac{r}{\sqrt{2rh}}$ and arrive at the same solution.

G. Under what conditions do the line of sight formula (question D) and the arc length formula (question E) give significantly different answers? When would you use the arc length formula on the Moon or on some other solar system body?

If r is comparable in size to *h*, there would be a significant difference. You would use the arc length formula on the Moon only if the antenna height were measured in hundreds of kilometers, which is not a reasonable engineering solution. You might, however, use the arc length formula on a small asteroid that is less than a few kilometers across.

Scoring Guide

Suggested 14 points total to be given.

Question		Distribution of points
A	*1 point*	1 point for the correct formula
B	*1 point*	1 point for the correct formula
C	*1 points*	1 point for the correct distance
D	*4 points*	1 point for the correct graph with correct labels and scale 1 point for explaining that distance increases at a decreasing rate 1 point for correct value of *ds/dh* 1 point for correct interpretation of *ds/dh*
E	*3 points*	1 point for correct linearization 1 point for comparison of linearization with actual value 1 point for correct conclusion that linearization is not as accurate for an antenna of height 11 m
F	*2 points*	1 point for correct derivative formula 1 point for correct numerical value
G	*2 points*	1 point for explaining when values would differ 1 point for explaining when to use the arc length formula

Contributors

Thanks to the subject matter experts for their contributions in developing this problem:

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