MATH AND SCIENCE @ WORK

AP* CALCULUS Educator Edition

SPACE SHUTTLE GUIDANCE, NAVIGATION, AND CONTROL DATA

Instructional Objectives

Students will

- gain an understanding of the M50 coordinate system;
- write a set of parametric equations based on a table of position coordinates over time; and
- differentiate parametric position functions to determine the velocity and acceleration vectors, as well as the magnitude of the velocity and acceleration.

Degree of Difficulty

This problem asks students to analyze a table of data and generate parametric functions based on the data. Students will be asked to explain why their answers may differ from the data.

- For the average Calculus BC student the problem may be at a moderate difficulty level.

Class Time Required

This problem requires 65-80 minutes.

- Introduction: 5 minutes
- Student Work Time: 40-60 minutes
- Post Discussion: 20 minutes

Background

This problem is part of a series of problems that apply Math and Science @ Work in NASA's Space Shuttle Mission Control Center.

Since its conception in 1981, NASA has used the space shuttle for human transport, the construction of the International Space Station (ISS), and to research the effects of space on the human body. One of the keys to the success of the Space Shuttle Program is the Space Shuttle Mission Control Center (MCC). The Space Shuttle MCC at NASA Johnson Space Center uses some of the most sophisticated technology and communication equipment in the world to monitor and control the space shuttle flights.

Grade Level
11-12

Key Topic
AP Calculus BC:
Parametric Equations

Degree of Difficulty
Moderate

Teacher Prep Time
10 minutes

Class Time Required
65-85 minutes

Technology
Graphing Calculator

AP Course Topics
Functions, Graphs, and Limits:
- Parametric, polar, and vector functions

Derivatives:
- Applications of derivatives
- Computation of derivatives

NCTM Principles and Standards
- Algebra
- Geometry
- Measurement
- Data Analysis and Probability
- Communication

Within the Space Shuttle MCC, teams of highly qualified engineers, scientists, doctors, and technicians, known as flight controllers, monitor the systems and activities aboard the space shuttle. They work together as a powerful team, spending many hours performing critical simulations as they prepare to support preflight, ascent, flight, and reentry of the space shuttle and the crew. The flight controllers provide the knowledge and expertise needed to support normal operations and any unexpected events.

One of the flight controllers in the Space Shuttle MCC is the Guidance, Navigation, and Control (GNC) officer. To understand the roles of the GNC officer, one must first understand the basics of the GNC system. Guidance equipment (gyroscopes and accelerometers) and software first compute the location of the vehicle and the orientation required to satisfy mission requirements. Navigation software then tracks the vehicle's actual location and orientation, allowing the flight controllers to use hardware to transport the space shuttle to the required location and orientation. The job of the GNC officer is to ensure the hardware and software that perform these functions are working correctly. This control portion of the process consists of two modes: automatic and manual. In the automatic mode, the primary avionics software system allows the onboard computers to control the guidance and navigation of the space shuttle. In the manual mode the crew uses data from the GNC displays and hand controls for the guidance and navigation. The GNC officer ensures that the GNC system has the accuracy and capacity necessary to control the space shuttle in both modes and that it is being utilized correctly.

The state vector of the space craft is the primary data used to determine the guidance function. The space shuttle's state vector is an estimate of vehicle position in space and velocity at a given time. Beginning with a known initial position, velocity, and orientation (such as on the launch pad just prior to launch), all sensed accelerations from that point can be integrated and incorporated with a physics model to calculate the new position, velocity, and orientation. For accurate control of the space craft the GNC officer must ensure that the state vector is accurate at all times during each mission phase (ascent, orbit operations, and reentry).

To understand how the state vector is calculated it is helpful to know the history involved in determining it. Throughout time, astronomers have used a three dimensional Cartesian coordinate system to identify positions in space. The origin of this coordinate system is located at the center of the Earth. The z-axis is defined as the line that runs through the North and South poles of the Earth. The x and y axis both lie on the plane formed by Earth's equator. The x-axis points toward the vernal equinox. Every year there are two equinoxes, one in the spring (the vernal equinox), and one in the fall (the autumnal equinox). An equinox occurs when the sun passes directly over the equator of the Earth causing equal amounts of daylight and night. The direction of the x-axis is always drifting because the Earth is always moving (rotating about the polar axis and orbiting the sun). For this reason it is necessary to fix the orientation of the x-axis at a particular moment in time. The M50 coordinate system (Figure 1) is based on the orientation of the Earth on January 1, 1950.

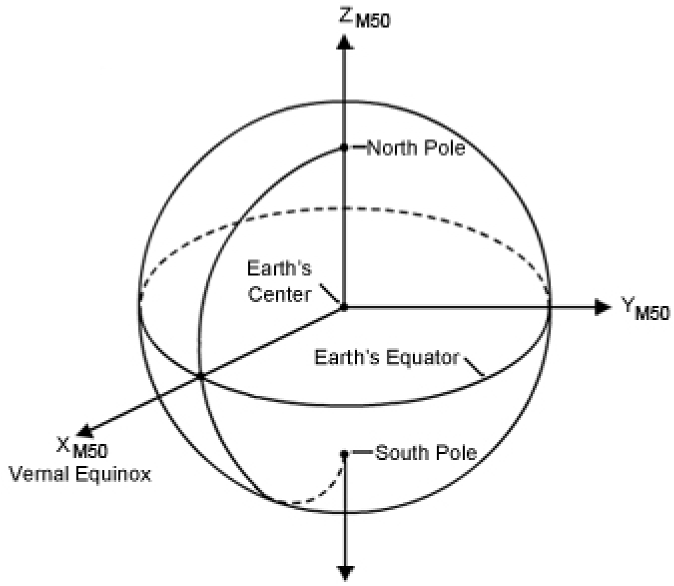

Figure 1: M50 (Aries-Mean-of1950) coordinate system

The space shuttle is equipped with three Inertial Measurement Units (IMUs) that are used for attitude and position estimation. These three IMUs are mounted on the navigation base that is a metal beam used to maintain a constant orientation with respect to the rest of the vehicle. (Figure 2)

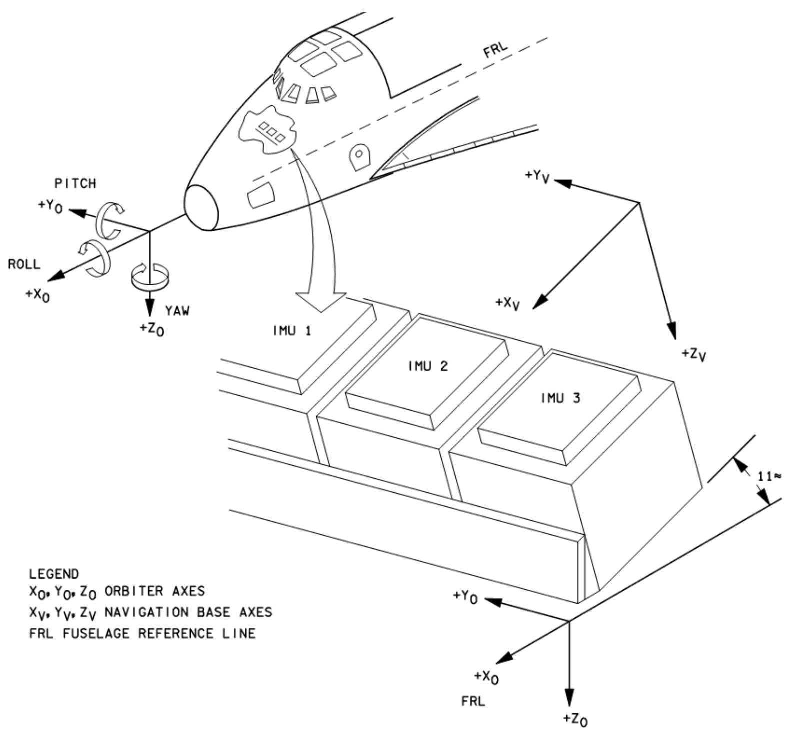

Figure 2: IMU position

At launch, the space shuttle's position, in M50 coordinates is known. The IMUs have accelerometers that measure acceleration in the *x*, *y*, and *z* directions, as defined by the space shuttle's frame of reference. By integrating the acceleration data, the IMUs can determine the space shuttle's velocity and position. This can then be used to determine the change in the initial M50 launch coordinates. The accuracy of this information is monitored by the GNC officer to ensure that the space shuttle arrives at its pre-determined destination as outlined by mission objectives.

AP Course Topics

Functions, Graphs, and Limits

- Parametric, polar, and vector functions
    - The analysis of planar curves includes those given in parametric form, polar form, and vector form

Derivatives

- Applications of derivatives
    - Analysis of planar curves given in parametric form, polar form, and vector form, including velocity and acceleration
- Computation of derivatives
    - Derivatives of parametric, polar, and vector functions

NCTM Principles and Standards

Algebra

- Understand patterns, relations, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Analyze change in various contexts

Geometry

- Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- Use visualization, spatial reasoning, and geometric modeling to solve problems

Measurement

- Understand measurable attributes of objects and the units, systems, and processes of measurement
- Apply appropriate techniques, tools, and formulas to determine measurements

Data Analysis and Probability

- Develop and evaluate inferences and predictions that are based on data

Communication

- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others

Problem

The table below gives M50 position and velocity data from the space shuttle accelerometers for approximately one orbit around the Earth. As you answer questions A – C, justify your answers by showing all work.

Time (minutes)	**Position (*x*)** (meters)	**Position (*y*)** (meters)	**Position (*z*)** (meters)	**Velocity (*x*)** $\left(\frac{\text{meters}}{\text{second}}\right)$	**Velocity (*y*)** $\left(\frac{\text{meters}}{\text{second}}\right)$	**Velocity (*z*)** $\left(\frac{\text{meters}}{\text{second}}\right)$
0	6521479	-734565	1458505	-732	5020	5798
5	5924596	784838	3078097	-3207	5010	4893
10	4635033	2212521	4336801	-5305	4415	3416
15	2804233	3381908	5087694	-6780	3305	1541
20	646291	4156853	5243442	-7464	1811	-513
25	-1587017	4447125	4786055	-7278	105	-2507
30	-3635191	4218696	3768833	-6242	-1613	-4208
35	-5258953	3497785	2310392	-4476	-3146	-5419
40	-6268174	2368178	581241	-2186	-4311	-5995
45	-6544580	961726	-1216040	361	-4973	-5868
50	-6055983	-557150	-2870787	2864	-5053	-5054
55	-4860024	-2010971	-4189288	5030	-4544	-3649
60	-3096963	-3230170	-5017592	6608	-3505	-1819
65	-972801	-4072773	-5259205	7414	-2058	224
70	1264767	-4440582	-4885923	7356	-370	2241
75	3354753	-4290364	-3941001	6440	1362	3997
80	5052847	-3639037	-2534456	4769	2938	5287
85	6159973	-2562141	-830686	2539	4171	5959
90	6546101	-1185374	970565	10	4917	5931

A. The actual duration of one orbit of the space shuttle is 90 minutes, 34 seconds. For simplicity assume the duration of one orbit is 90 minutes, and use the data given to write a set of parametric equations that approximate the position of the space shuttle at any time, t, where t is in minutes and position is in meters.

Note: *An orbital motion should have x, y, and z coordinates that oscillate between minimum and maximum values. This suggests sinusoidal functions. For simplicity, ignore the drag and precessional forces caused by the Sun and Moon and use a simple sinusoidal function of the form:*

$$f(t) = A\sin(k(t-\phi)) + V$$

I. Find the position of the space shuttle in meters in M50 coordinates according to your equations when $t = 15$.

II. How does your answer compare to the accelerometer data for $t = 15$? Can you explain any difference in your predicted position and accelerometer data?

B. Write a set of parametric equations that will give the space shuttle's velocity at any time, *t*.

I. Use your equations to find the velocity vector of the space shuttle in meters per second when $t = 10$ and compare it to the table.

II. Find the speed of the space shuttle in meters per second when $t = 10$.

C. Write a set of parametric equations that will give the space shuttle's acceleration at any time, *t*.

I. Use your equations to find the acceleration vector of the space shuttle in meters per second squared when $t = 10$.

II. Find the magnitude of the space shuttle's acceleration in meters per second squared when $t = 10$. Do you notice anything about your answers? (Hint: Why do astronauts experience weightlessness when they are still within the gravitational field of the Earth?)

Solution Key (One Approach)

A. The actual duration of one orbit of the space shuttle is 90 minutes, 34 seconds. For simplicity assume the duration of one orbit is 90 minutes, and use the data given to write a set of parametric equations that approximate the position of the space shuttle at any time, *t*, where *t* is in minutes and position is in meters.

Note: *An orbital motion should have x, y, and z coordinates that oscillate between minimum and maximum values. This suggests sinusoidal functions. For simplicity, ignore the drag and precessional forces caused by the Sun and Moon and use a simple sinusoidal function of the form:*

$$f(t) = A\sin(k(t-\phi)) + V$$

Upon inspection of the data students may notice that the space shuttle does not return to its initial position when it completes one orbit. There are a couple of reasons for this. First, the actual period of one orbit is slightly longer than 90 minutes (91 minutes, 34 seconds). In addition, the space shuttle experiences forces from atmospheric drag due to the rarefied atmosphere at the orbital altitude and orbital precession caused by the gravitational effects of the Sun and Moon on the spacecraft. That means the point from which the space shuttle has started an orbit has moved during the 90 minutes. Therefore, at the end of one orbit the space shuttle returns to a slightly different point.

If the period of one orbit is taken as 90 minutes, then the coefficient *k* is found by:

$$90 = \frac{2\pi}{k}$$

$$k = \frac{\pi}{45}$$

This coefficient will be the same for the *x*, *y*, and *z* functions since they all have the same period.

Inspection of the *x* coordinate data suggests a maximum value of 6,546,101 meters and a minimum of -6,544,580 meters. These can be used to define the amplitude, *A*, and the vertical shift, *V*:

$$A = \frac{\max - \min}{2}$$

$$A = \frac{6546101 - (-6544580)}{2}$$

$$A = 6545340.5$$

$$V = \max - amplitude$$

$$V = 6546101 - 6545340.5$$

$$V = 760.5$$

The phase shift could be found by substituting in the values found thus far, setting $t = 0$, and solving for ϕ, or by simply noticing which value of *t* the assumed maximum value occurs at. For a normal sine curve the maximum should occur at the end of the first quarter period. Since the period of one orbit is 90 minutes, the first quarter period ends when $t = 22.5$. For the *x* data, the maximum occurs when $t = 90$, thus this curve has shifted 67.5 minutes to the right, or 22.5 minutes to the left.

Substituting all of these values into the basic equation, the *x*(*t*) function, looks like:

$$x(t) = 6545340.5\sin\left(\frac{\pi}{45}(t + 22.5)\right) + 760.5$$

Using the same methodology for the *y* and *z* equations, the full set of parametric equations looks like:

$$x(t) = 6545340.5\sin\left(\frac{\pi}{45}(t + 22.5)\right) + 760.5$$

$$y(t) = 4443853.5\sin\left(\frac{\pi}{45}(t - 2.5)\right) + 3271.5$$

$$z(t) = 5251323.5\sin\left(\frac{\pi}{45}(t + 2.5)\right) - 7881.5$$

I. Find the position of the space shuttle in meters in M50 coordinates according to your equations when $t = 15$.

Evaluating these equations at $t = 15$ yields:

x(15) = 3273430.750 meters

y(15) = 3407460.780 meters

z(15) = 4926748.442 meters

II. How does your answer compare to the accelerometer data for $t = 15$? Can you explain any difference in your predicted position and accelerometer data?

x(15) = 3273430.750 meters	Actual = 2804233 meters
y(15) = 3407460.780 meters	Actual = 3381908 meters
z(15) = 4926748.442 meters	Actual = 5087694 meters

Difference may be due to the difference in the 90 minute period and the actual period of 90 minutes and 34 seconds, and the failure to account for the procession of the Earth around the Sun.

B. Write a set of parametric equations that will give the space shuttle's velocity at any time, *t*.

Velocity is the derivative of position with respect to time. Differentiating each of the position equations with respect to *t* yields:

$$x'(t) = 6545340.5\cos\left(\frac{\pi}{45}(t+22.5)\right)\cdot\frac{\pi}{45} = 456950.970\cos\left(\frac{\pi}{45}(t+22.5)\right)$$

$$y'(t) = 4443853.5\cos\left(\frac{\pi}{45}(t-2.5)\right)\cdot\frac{\pi}{45} = 310239.500\cos\left(\frac{\pi}{45}(t-2.5)\right)$$

$$z'(t) = 5251323.5\cos\left(\frac{\pi}{45}(t+2.5)\right)\cdot\frac{\pi}{45} = 366611.541\cos\left(\frac{\pi}{45}(t+2.5)\right)$$

I. Use your equations to find the velocity vector of the space shuttle in meters per second when $t = 10$ and compare it to the table.

Evaluating these equations at $t = 10$ yields:

$$x'(t) = -293722.422\,\frac{\text{m}}{\text{min}}$$

$$y'(t) = 268675.288\,\frac{\text{m}}{\text{min}}$$

$$z'(t) = 235653.356\,\frac{\text{m}}{\text{min}}$$

Note that since the position equations were written in terms of *t* in minutes, these velocities are in meters per minute. To make a clearer comparison to the table of data, which is in meters per second, each value should be divided by 60.

$$x'(t) = -4895.374\,\frac{\text{m}}{\text{s}}$$

$$y'(t) = 4477.921\,\frac{\text{m}}{\text{s}}$$

$$z'(t) = 3927.556\,\frac{\text{m}}{\text{s}}$$

II. Find the speed of the space shuttle in meters per second when $t = 10$.

Speed is the magnitude of the velocity vector. Magnitude is found in a manner similar to the distance formula:

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(-4895.374)^2 + (4477.921)^2 + (3927.556)^2}$$

$$\text{Speed} = 7709.874\,\frac{\text{m}}{\text{s}}$$

C. Write a set of parametric equations that will give the space shuttle's acceleration at any time, *t*.

Acceleration is the derivative of velocity with respect to time. Differentiating each of the velocity equations with respect to *t* yields:

$$x''(t) = -456950.970\sin\left(\frac{\pi}{45}(t+22.5)\right)\cdot\frac{\pi}{45} = -31901.196\sin\left(\frac{\pi}{45}(t+22.5)\right)$$

$$y''(t) = -310239.500\sin\left(\frac{\pi}{45}(t-2.5)\right)\cdot\frac{\pi}{45} = -21658.803\sin\left(\frac{\pi}{45}(t-2.5)\right)$$

$$z''(t) = -366611.541\sin\left(\frac{\pi}{45}(t+2.5)\right)\cdot\frac{\pi}{45} = -25594.314\sin\left(\frac{\pi}{45}(t+2.5)\right)$$

I. Use your equations to find the acceleration vector of the space shuttle in meters per second squared when $t = 10$.

Evaluating these equations at $t = 10$ yields:

$$x''(t) = -24437.734\,\frac{\text{m}}{\text{min}^2}$$

$$y''(t) = -10829.402\,\frac{\text{m}}{\text{min}^2}$$

$$z''(t) = -19606.382\,\frac{\text{m}}{\text{min}^2}$$

Because the position functions were written in terms of *t* in minutes, the acceleration vector is in terms of meters per minute squared. To convert to meters per second squared, divide by 3600.

$$x''(t) = -6.788\,\frac{\text{m}}{\text{s}^2}$$

$$y''(t) = -3.008\,\frac{\text{m}}{\text{s}^2}$$

$$z''(t) = -5.446\,\frac{\text{m}}{\text{s}^2}$$

II. Find the magnitude of the space shuttle's acceleration in meters per second squared when $t = 10$. What do you notice about your answer that might explain why astronauts experience weightlessness when they are still within the gravitational field of the Earth? (Hint: think about what you know about gravity and acceleration.)

Once again, magnitude of the vector is found by taking the square root of the sum of the squares of each component:

$$|a| = \sqrt{(-6.788)^2 + (-3.008)^2 + (-5.446)^2}$$

$$|a| = 9.208\,\frac{\text{m}}{\text{s}^2}$$

Note how close this is to the normal acceleration due to gravity, 9.8 m/s^2. Astronauts experience weightlessness because they are actually in a constant free fall, accelerating at roughly the same rate as gravity. Because the position equations were not exact, the acceleration is not exact. Furthermore, the acceleration due to gravity decreases as altitude increases. If you adjust the acceleration due to gravity for the altitude of the space shuttle in orbit, then the magnitude of the acceleration calculated is even closer to the acceleration due to gravity.

Scoring Guide

Suggested 15 points total to be given.

Question		Distribution of points
A	*5 points*	3 points (1 point each) for the parametric equations $x(t)$, $y(t)$, $z(t)$
		1 point for the correct answers to part I
		1 point for the correct answer and explanation to part II
B	*5 points*	3 points (1 point each) for the parametric equations $x'(t)$, $y'(t)$, $z'(t)$
		1 point for the correct answers to part I
		1 point for the correct answer to part II
C	*5 points*	3 points (1 point each) for the parametric equations $x''(t)$, $y''(t)$, $z''(t)$
		1 point for the correct answers to part I
		1 point for the correct answer to part II

Contributors

This problem was developed by the Human Research Program Education and Outreach (HRPEO) team with the help of NASA subject matter experts and high school AP instructors.

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