MATH AND SCIENCE @ WORK

AP* CALCULUS Educator Edition

SPACE SHUTTLE AUXILIARY POWER UNITS

Instructional Objectives

Students will

- analyze graphs to determine the rate of change at specific points; and
- use the chain rule to determine the rate of change of two or more variables that are changing with respect to time.

Degree of Difficulty

This problem is an application of various calculus concepts including an application of related rates. The focus is on interpretation of the derivative as a rate of change.

- For the average student in Calculus AB and Calculus BC, the problem may be of basic difficulty.

Class Time Required

More outside science knowledge is referred to in this problem than would be typical on an AP free response question**.** Class discussion is encouraged allowing for students to make connections between calculus and other courses of study, such as chemistry. This problem requires 55-70 minutes.

- Introduction: 15-20 minutes
- Student Work Time: 20-25 minutes
- Post Discussion: 20-25 minutes

Background

This problem is part of a series of problems that apply Math and Science @ Work in NASA's Space Shuttle Mission Control Center.

Since its conception in 1981, NASA has used the space shuttle for human transport, the construction of the International Space Station (ISS), and to research the effects of space on the human body. One of the keys to the success of the Space Shuttle Program is the Space Shuttle Mission Control Center (MCC). The Space Shuttle MCC at NASA Johnson Space Center uses some of the most sophisticated technology and communication equipment in the world to monitor and control the space shuttle flights.

Grade Level
10-12

Key Topic
Application of differentiation - related rates

Degree of Difficulty
Calculus AB: Basic
Calculus BC: Basic

Teacher Prep Time
30 minutes

Class Time Required
55-70 minutes

Technology
Calculator (graphing preferable)

AP Course Topics
Functions, Graphs and Limits:
- Analysis of graphs

Derivatives:
- Concept of the derivative
- Derivative at a point
- Applications of derivative
- Computation of derivative

NCTM Principles and Standards
- Algebra
- Measurement
- Communication
- Representation

Within the Space Shuttle MCC, teams of highly qualified engineers, scientists, doctors, and technicians, known as flight controllers, monitor the systems and activities aboard the space shuttle. They work together as a powerful team, spending many hours performing critical simulations as they prepare to support preflight, ascent, flight, and reentry of the space shuttle and the crew. The flight controllers provide the knowledge and expertise needed to support normal operations and any unexpected events.

One member of this team of experts is the Mechanical, Maintenance, Arm, and Crew Systems flight controller, whose call sign is MMACS (pronounced “Max”). One of the responsibilities of this position is to monitor the performance and fuel usage of the space shuttle’s Auxiliary Power Units (APUs). Three identical but independent hydraulic systems are used to assist in maneuvering the space shuttle, deploying the landing gear, and applying the brakes. Each of these hydraulic systems is powered by a separate APU. The APUs convert chemical energy into mechanical shaft power to drive the hydraulic main pumps. The chemical energy is in the form of hydrazine fuel (N_2H_4), which is stored in three independent tanks. The location of the APUs and fuel tanks can be seen in Figure 1.

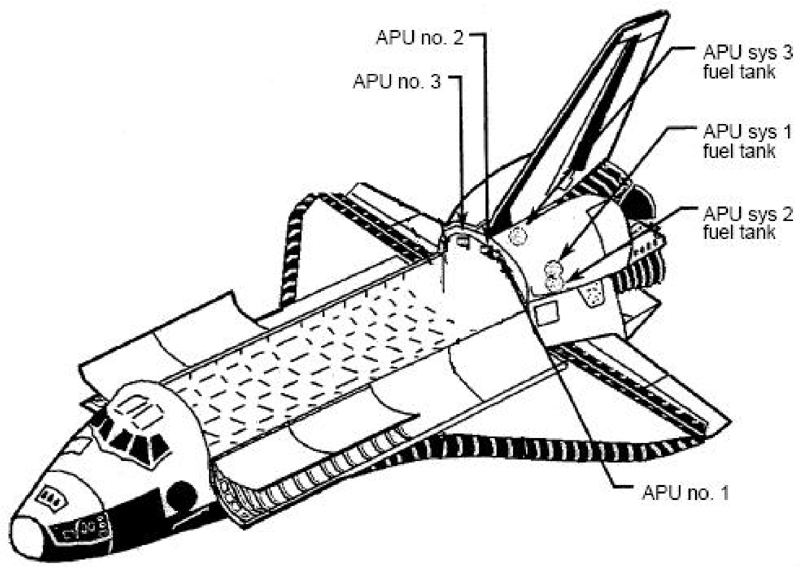


Figure 1: Space shuttle with APU and fuel tank locations

The MMACS flight controller monitors the fuel usage of the APUs by looking at plots of fuel tank pressure versus time, as well as fuel quantity versus time. The fuel quantity is not directly measured, but is calculated using the fuel tank pressure and a number of constants.

AP Course Topics

Functions, Graphs and Limits
- Analysis of Graphs

Derivatives
- Concept of the derivative
    - Derivative interpreted as an instantaneous rate of change
- Derivative at a point
    - Slope of a curve at a point
    - Approximate rate of change from graphs and tables of values
- Applications of derivatives
    - Modeling rates of change, including related rates problems

  - Interpretation of the derivative as a rate of change in varied applied contexts
- Computation of derivatives
  - Chain rule

NCTM Principles and Standards

Algebra
- Understand patterns, relations, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Analyze change in various contexts

Measurement
- Apply appropriate techniques, tools, and formulas to determine measurements

Communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Use the language of mathematics to express mathematical ideas precisely

Representation
- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems

Problem

When the fuel tank valves are opened and the Auxiliary Power Unit (APU) is started, the fuel, hydrazine (N_2H_4), is forced out of the tank using gaseous nitrogen (GN_2) at high pressures. A schematic of a fuel tank and APU can be seen in Figure 2. As the APU burns fuel, the mass of fuel in the tank decreases, while the mass of nitrogen remains constant. The pressure of the fuel tank is directly measured and used to calculate the quantity of fuel.

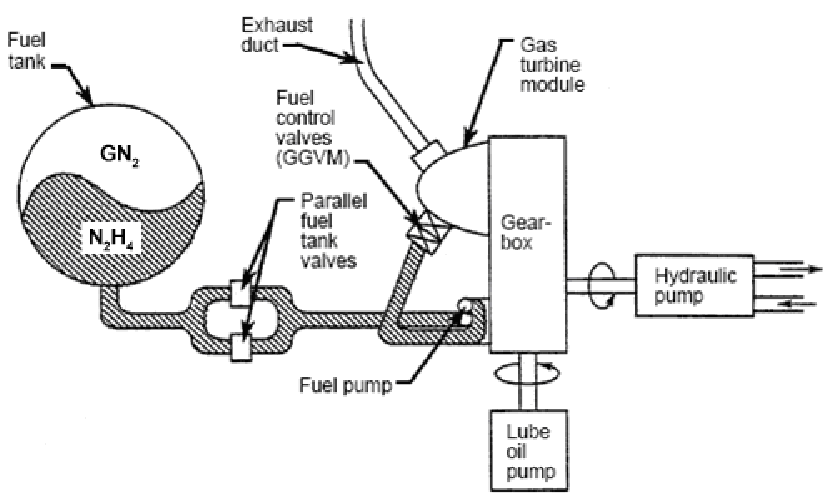


Figure 2: Schematic of a fuel tank and APU

The list of variables and constants is provided in Table 1 and will prove beneficial in solving the following questions.

Table 1: Variables and constants used throughout the problem

V_{N_2} = Volume of Nitrogen (N_2)	changing variable
$m_{N_2H_4}$ = Mass of Hydrazine (N_2H_4)	changing variable
P = Pressure (psi)	changing variable
V_{tank} = Volume of Tank	constant = 11,494 in^3
m_{N_2} = Mass of Nitrogen (N_2)	constant = 2.688 lbs
Rho = density of Hydrazine (N_2H_4)	constant = 0.0363 $\frac{lbs}{in^3}$
T = Temperature (Rankin)	constant = 530 ^oR
R = Universal Gas	constant = 661.8 $\frac{in \cdot lbs}{lbs \cdot {}^oR}$

A. Assume you are the MMACS flight controller and your control panel shows the following fuel tank pressure vs. time plot (figure 3).

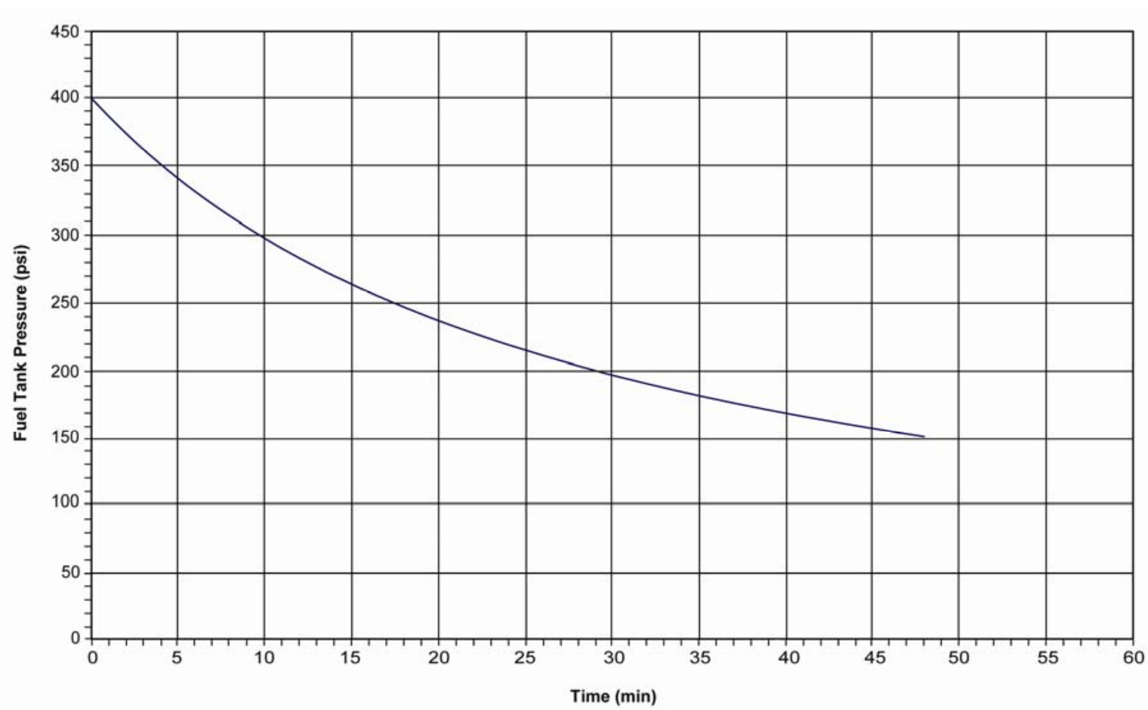


Figure 3: Fuel Tank Pressure vs. Time

I. Use the graph to explain what is happening to the fuel tank pressure over time.

II. Find an appropriate interval around 1 minute and estimate the average rate of change of the fuel tank pressure. What is the average rate of change of the pressure around 15 minutes? Around 35 minutes?

III. Explain the meaning of the average rates of change just found.

IV. How would your estimated average rates of change compare to the instantaneous rates of change or $\frac{dP}{dt}$ at each point? Explain your reasoning.

B. Fuel mass is a combination of the fuel's density, fuel pressure at a specific time, and the volume of the tank. It is computed using a formula that relates the quantities in the following three equations.

$$V_{N_2} = \frac{m_{N_2} \cdot R \cdot T}{P} \quad \text{(ideal gas law for } V_{N_2}\text{)}$$

$$V_{tank} = V_{N_2} + V_{N_2H_4} \quad \text{(tank volume balance)}$$

$$m_{N_2H_4} = V_{N_2H_4} \cdot Rho \quad \text{(the mass fuel density equation for incompressible fluid)}$$

Use algebra to derive the equation for the mass of hydrazine:

$$m_{N_2H_4} = Rho \cdot V_{tank} - \frac{Rho \cdot m_{N_2} \cdot R \cdot T}{P}$$

C. Use the equation derived in question B to help answer the questions below.

I. Complete the table below by calculating the mass of hydrazine in the fuel tank. Use Figure 3 to find pressure at a given time.

Time (min)	Pressure (psi)	$m_{N_2H_4}$ (lbs)
1		
15		
35		

II. Use your values to sketch a linear graph of Fuel Quantity ($m_{N_2H_4}$) vs. Time and estimate the rate of change of the line. If a graphing calculator is available, use the capabilities of the calculator to plot the points and to find the slope.

D. Differentiate the equation for mass of hydrazine, $m_{N_2H_4} = Rho \cdot V_{tank} - \frac{Rho \cdot m_{N_2} \cdot R \cdot T}{P}$, with respect to time, t, to determine the fuel usage rate. Remember the only variables that are changing are mass of hydrazine, $m_{N_2H_4}$, and pressure, P. Explain the practical meaning of the derivative with units.

E. The solutions found in questions A – D will help you answer the following.

I. Complete the following table using the rate of change formula from question D to calculate the rate of change of hydrazine ($m_{N_2H_4}$). Use Figure 3 and your answers from question A when completing the table.

Time (min)	Pressure (psi)	$\frac{dP}{dt}$ (psi/min)	$\frac{dm_{N_2H_4}}{dt}$ (lbs/min)
1			
15			
35			

II. How does this calculated rate of change of hydrazine $\left(\frac{dm_{N_2H_4}}{dt}\right)$ compare to the slope of the graph of Fuel Quantity ($m_{N_2H_4}$) vs. Time found in question C, part II?

F. As the MMACS flight controller, you notice an unusual trend in the fuel tank pressure vs. time plot indicating a possible leak. Since it is difficult to troubleshoot a leak while an APU is in operation, Mission Control will have the crew on board the space shuttle shut one APU down. Figure 4 represents the data from the APU.

I. Based on the graph (Figure 4), how would you determine if there is a leak in the fuel tank?

II. If there is a leak, how fast is the leak rate in lbs/min at t = 50 minutes?

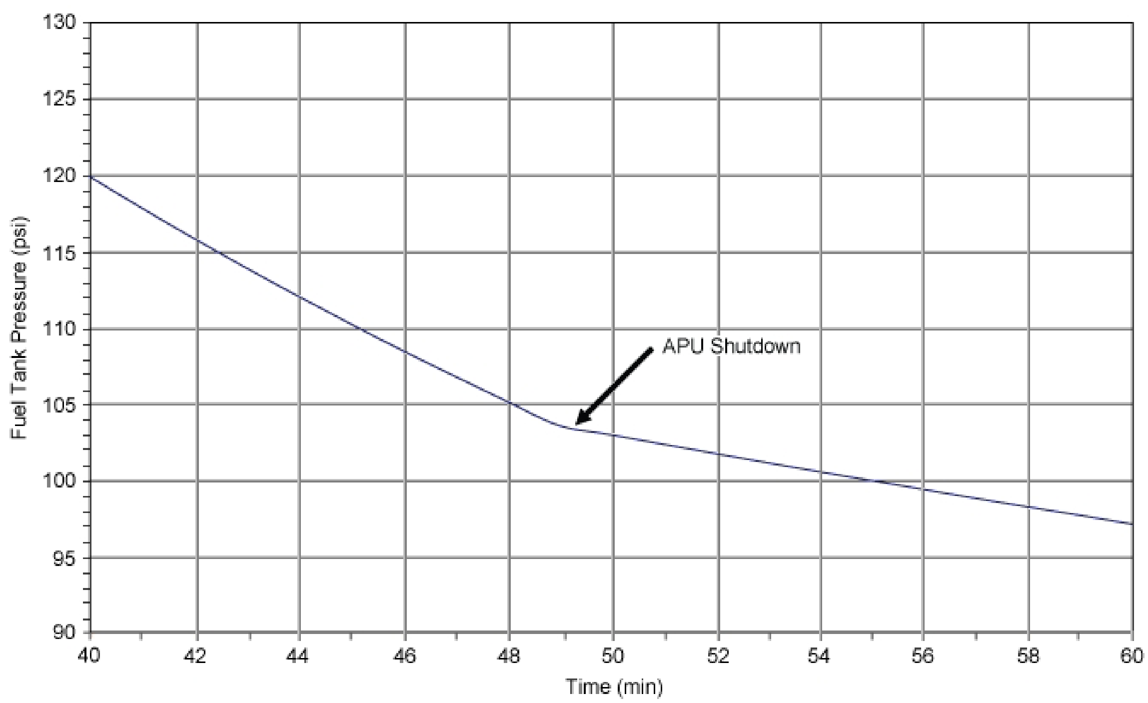


Figure 4: Fuel Tank Pressure vs. Time

Solution Key (One Approach)

A. Assume you are the MMACS flight controller and your control panel shows the following Fuel Tank Pressure vs. Time plot (figure 3).

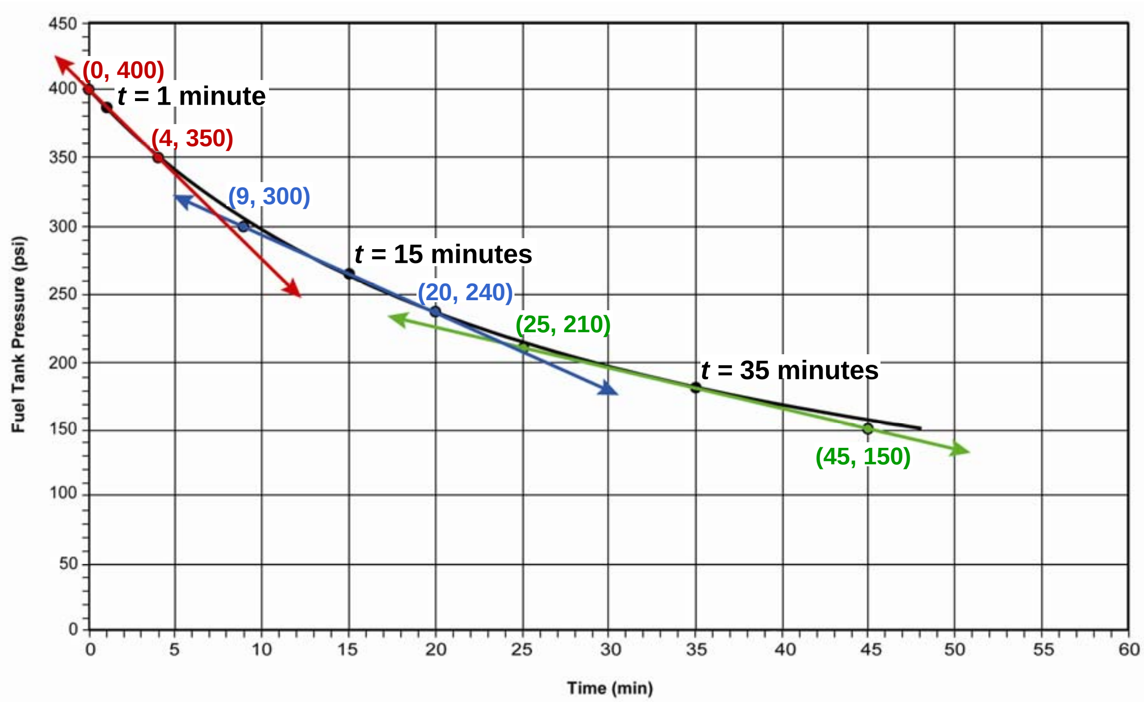
I. Use the graph to explain what is happening to the fuel tank pressure over time.

The fuel tank pressure is decreasing over time. The rate of decrease slows as time passes.

II. Find an appropriate interval around 1 minute and estimate the average rate of change of the fuel tank pressure. What is the average rate of change of the pressure around 15 minutes? Around 35 minutes?

Answers will vary.

Students should draw tangent lines to approximate the slopes or use readable points on the graph for approximations.



At 1 minute: $\dfrac{400-350}{0-4}=\dfrac{50}{-4}=-12.500$ psi/min

At 15 minutes: $\dfrac{300-240}{9-20}=\dfrac{60}{-11}=-5.455$ psi/min

At 35 minutes: $\dfrac{210-150}{25-45}=\dfrac{60}{-20}=-3.000$ psi/min

III. Explain the meaning of the average rates of change just found.

The rates of change found indicate that at 1 minute the fuel tank pressure is decreasing by 12.500 psi every minute; at 15 minutes it is decreasing by 5.455 psi per minute; at 35 minutes it is decreasing by 3.000 psi per minute. This verifies the observation from part I that the rate of decrease slows as time passes.

IV. How would your estimated average rates of change compare to the instantaneous rates of change or $\frac{dP}{dt}$ at each point? Explain your reasoning.

They should be very similar if not exactly the same. The instantaneous rate of change or $\frac{dP}{dt}$ is the slope of the tangent line at a particular point. Since we estimated the average rates of change they may be slightly different from the calculated instantaneous rates of change.

B. Fuel mass is a combination of the fuel's density, fuel pressure at a specific time, and the volume of the tank. It is computed using a formula that relates the quantities in the following three equations.

$$V_{N_2} = \frac{m_{N_2} \cdot R \cdot T}{P} \text{ (ideal gas law for } V_{N_2}\text{)} \quad (1)$$

$$V_{tank} = V_{N_2} + V_{N_2H_4} \text{ (tank volume balance)} \quad (2)$$

$$m_{N_2H_4} = V_{N_2H_4} \cdot Rho \text{ (the mass fuel density equation for incompressible fluid)} \quad (3)$$

Use algebra to derive the equation for the mass of hydrazine:

$$m_{N_2H_4} = Rho \cdot V_{tank} - \frac{Rho \cdot m_{N_2} \cdot R \cdot T}{P} \quad (4)$$

Substitute equation (1) into equation (2) to obtain: $V_{tank} = \frac{m_{N_2} \cdot R \cdot T}{P} + V_{N_2H_4}$ (5)

Divide both sides of equation (3) by *Rho* to obtain: $\frac{m_{N_2H_4}}{Rho} = V_{N_2H_4}$ (6)

Substitute equation (6) into equation (5) to obtain:

$$V_{tank} = \frac{m_{N_2} \cdot R \cdot T}{P} + \frac{m_{N_2H_4}}{Rho} \quad (7)$$

Multiply both sides of equation (7) by *Rho* to obtain:

$$Rho \cdot V_{tank} = \frac{Rho \cdot m_{N_2} \cdot R \cdot T}{P} + m_{N_2H_4} \quad (8)$$

Solve equation (8) for $m_{N_2H_4}$ to obtain equation (4).

C. Use the equation derived in question B to help answer the questions below.

I. Complete the table below by calculating the mass of hydrazine in the fuel tank. Use Figure 3 to find pressure at a given time.

Using equation (4), substitute the given constants' values to obtain:

$$m_{\mathrm{N_2H_4}} = Rho \cdot V_{\mathrm{tank}} - \frac{Rho \cdot m_{\mathrm{N_2}} \cdot R \cdot T}{P}$$

$$m_{\mathrm{N_2H_4}} = \left(0.0363\,\tfrac{\mathrm{lbs}}{\mathrm{in}^3}\right) \cdot \left(11{,}494\ \mathrm{in}^3\right) - \frac{\left(0.0363\,\tfrac{\mathrm{lbs}}{\mathrm{in}^3}\right) \cdot (2.688\ \mathrm{lbs}) \cdot \left(661.8\,\tfrac{\mathrm{in}\cdot\mathrm{lbs}}{\mathrm{lbs}\cdot{}^\circ\mathrm{R}}\right) \cdot (530\ {}^\circ\mathrm{R})}{P}$$

$$m_{\mathrm{N_2H_4}} = 417.232\ \mathrm{lbs} - \frac{34224.611\,\tfrac{\mathrm{lbs}^2}{\mathrm{in}^2}}{P} \tag{9}$$

Complete the chart by obtaining pressure values from the graph given in question A and substituting those into equation (9) to determine the mass values for hydrazine.

Answers will vary.

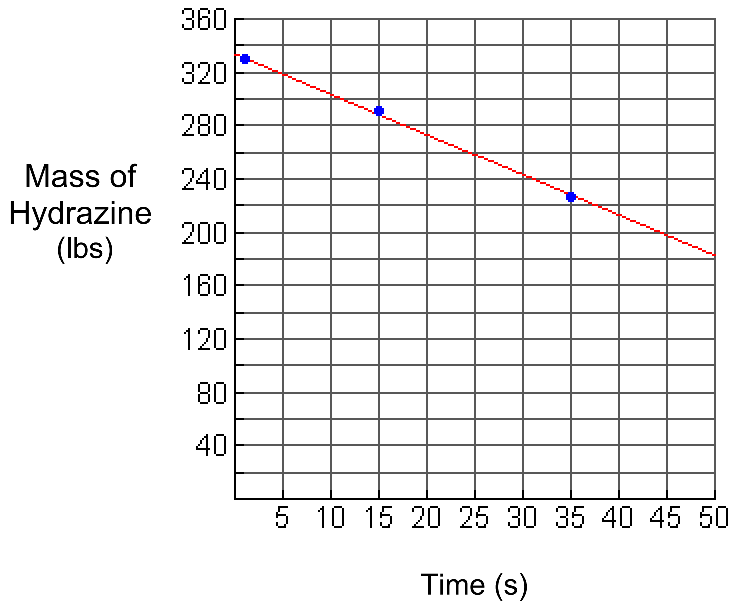
Time (min)	Pressure (psi)	$m_{\mathrm{N_2H_4}}$ (lbs)
1	390	329.477
15	270	290.474
35	180	227.095

II. Use your values to sketch a linear graph of Fuel Quantity ($m_{\mathrm{N_2H_4}}$) vs. Time and estimate the rate of change of the line. If a graphing calculator is available, use the capabilities of the calculator to plot the points and to find the slope.

Plot the three ordered pairs of (time, $m_{\mathrm{N_2H_4}}$) and sketch a line of best fit.

Points and graphs will vary.

Fuel Quantity ($m_{N_2H_4}$) vs. Time



Approximate the slope by using two points on the line of best fit. Answers will vary.

$$\frac{320-260}{5-25} = -3.000\,\frac{\text{lbs}}{\text{min}}$$

Using graphing calculators, students may determine the linear regression to find the slope. With the points given above they would find a slope of $-3.022\,\frac{\text{lbs}}{\text{min}}$.

D. Differentiate the equation for mass of hydrazine, $m_{N_2H_4} = Rho \cdot V_{\text{tank}} - \dfrac{Rho \cdot m_{N_2} \cdot R \cdot T}{P}$, with respect to time, t, to determine the fuel usage rate. Remember the only variables that are changing are mass of hydrazine, $m_{N_2H_4}$, and pressure, P. Explain the practical meaning of the derivative with units.

Begin with equation (4) from question B and differentiate with respect to time using the chain rule:

$$m_{N_2H_4} = Rho \cdot V_{\text{tank}} - \frac{Rho \cdot m_{N_2} \cdot R \cdot T}{P}$$

$$\frac{dm_{N_2H_4}}{dt} = \frac{Rho \cdot m_{N_2} \cdot R \cdot T}{P^2} \cdot \frac{dP}{dt} \tag{10}$$

Substitute the constant values to obtain:

$$\frac{dm_{N_2H_4}}{dt} = \frac{\left(0.0363\,\frac{\text{lbs}}{\text{in}^3}\right) \cdot \left(2.688\text{ lbs}\right) \cdot \left(661.8\,\frac{\text{in}\cdot\text{lbs}}{\text{lbs}\cdot{}^\circ\text{R}}\right) \cdot \left(530\,{}^\circ\text{R}\right)}{P^2} \cdot \frac{dP}{dt} \tag{11}$$

$$\frac{dm_{N_2H_4}}{dt} = \frac{34224.611 \frac{\text{lbs}^2}{\text{in}^2}}{P^2} \cdot \frac{dP}{dt} \qquad (12)$$

This derivative represents the rate of change of the mass of hydrazine, $m_{N_2H_4}$, in lbs per minute. Knowing how much fuel is used per minute to power the APU would help to determine how much is needed for a mission.

E. The solutions found in questions A – D will help you answer the following.

I. Complete the following table using the rate of change formula from question D to calculate the rate of change of hydrazine ($m_{N_2H_4}$). Use Figure 3 and your answers from question A when completing the table.

Substitute the pressure and the rate of change from question A into equation (12) to calculate the rate of change of hydrazine.

Time (min)	Pressure (psi)	$\frac{dP}{dt}$ (psi/min)	$\frac{dm_{N_2H_4}}{dt}$ (lbs/min)
1 390		-12.500	-2.813
15 265		-5.455	-2.659
35 180		-3.000	-3.169

II. How does this calculated rate of change of hydrazine $\left(\frac{dm_{N_2H_4}}{dt}\right)$ compare to the slope of the graph of Fuel Quantity ($m_{N_2H_4}$) vs. Time found in question C, part II?

The calculated rate of change of hydrazine approaches the linear approximation as time increases.

F. As the MMACS flight controller, you notice an unusual trend in the fuel tank pressure vs. time plot indicating a possible leak. Since it is difficult to troubleshoot a leak while an APU is in operation, Mission Control will have the crew on board the space shuttle shut one APU down. Figure 4 represents the data from the APU.

I. Based on the graph, how would you determine if there is a leak in the fuel tank?

Prior to shutdown, the fuel tank pressure is decreasing for two reasons: (1) because the APU is burning fuel to convert chemical energy into mechanical energy and (2) there is a leak. When the APU is shut down, the APU is no longer burning, so when the MMACS flight controller observes that the fuel tank pressure is still decreasing, they know that it must be due to a leak.

Note to Educator: There are different reasons that an APU might leak. This could include an improperly connected fuel line fitting, a Micro-Meteoroid impact, an explosion near *the APU, stress on the APU fuel lines, or improper workmanship during ground processing.*

II. If there is a leak, how fast is the leak rate in lbs/min at t = 50 minutes?

$\frac{dP}{dt} = \frac{103 - 100}{50 - 55} = -\frac{3}{5} = -0.6\,\frac{\text{psi}}{\text{min}}$, with pressure approximately 103 psi.

Note to teacher: Discuss with students what $\frac{\text{psi}}{\text{min}}$ *means and how it can also be written as* $\frac{\text{lbs}}{\text{in}^2 \cdot \text{min}}$*. This second way of writing it can help determine correct units when substituting the values into an equation.*

Substitute into equation (12) from question D.

$$\frac{dm_{N_2H_4}}{dt} = \frac{34224.611\,\frac{\text{lbs}^2}{\text{in}^2}}{P^2} \cdot \frac{dP}{dt}$$

$$\frac{dm_{N_2H_4}}{dt} = \frac{34224.611\,\frac{\text{lbs}^2}{\text{in}^2}}{\left(103\,\frac{\text{lbs}}{\text{in}^2}\right)^2} \cdot \left(-0.6\,\frac{\text{lbs}}{\text{in}^2 \cdot \text{min}}\right)$$

$$\frac{dm_{N_2H_4}}{dt} = \frac{34224.6111}{(103)^2} \cdot (-0.6)$$

$$\frac{dm_{N_2H_4}}{dt} = -1.936\,\frac{\text{lbs}}{\text{min}}$$

Scoring Guide

Suggested 14 points total to be given.

Question		Distribution of points
A	*4 points*	1 point for a reasonable explanation in part I. 1 point for three reasonable estimates in part II. 1 point for a reasonable explanation in part III. 1 point for a correct explanation in part IV.
B	*1 point*	1 point for correct validation.
C	*2 points*	1 point for correct values in chart for part I. 1 point for regression line and slope in part II.
D	*2 points*	1 point for correct differentiation. 1 point for explaining the practical meaning with units.
E	*3 points*	1 point for correct pressures and $\frac{dP}{dt}$ values in part I. 1 point for correct $\frac{dm_{N_2H_4}}{dt}$ values in chart in part I. 1 point for correct comparison of $\frac{dm_{N_2H_4}}{dt}$ values to the slope in part II.
F	*2 points*	1 point for reasonable explanation in part I. 1 point for correct rate in part II.

Contributors

This problem was developed by the Human Research Program Education and Outreach (HRPEO) team with the help of NASA subject matter experts and high school AP instructors.

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