



MATH AND SCIENCE @ WORK

AP* CALCULUS Educator Edition



NEXT GENERATION SPACECRAFT

Instructional Objectives

Students will

- use integration to find the volume of a solid generated by a region, R ;
- determine the equation of a circle using the standard form and the general form;
- determine the equation of a line using the point-slope form; and
- solve a system of equations with three equations and three unknowns.

Degree of Difficulty

This problem is very challenging because students need to recall and apply many mathematical concepts from Algebra I.

- For the average AP Calculus AB student the problem may be very difficult (advanced).
- For the average AP Calculus BC student the problem may be moderately difficult.

If you are short on time or would like to simplify the problem for AP Calculus AB students, one option is to provide the students with the Computer Aided Drawing (CAD) diagram that includes the piecewise function without the limits of integration (see page 11). The students will then determine the limits of integration and select the appropriate application of integration. Doing so significantly reduces the amount of time needed to complete the problem.

Background

This problem is part of a series of problems that apply math and science principles to human space exploration at NASA.

Exploration provides the foundation of our knowledge, technology, resources, and inspiration. It seeks answers to fundamental questions about our existence, responds to recent discoveries, and puts in place revolutionary techniques and capabilities to inspire our nation, the world, and the next generation. Through NASA, we touch the unknown, we learn and we understand. As we take our first steps toward sustaining a human

Grade Level

11-12

Key Topic

Application of definite integrals – determining volume of a region

Degree of Difficulty

Calculus AB: Advanced
Calculus BC: Moderate

Teacher Prep Time

5 minutes

Problem Duration

45-60 minutes

Technology

Graphing Calculator

Materials

Student Edition including:

- Background handout
- Problem worksheet
- Support diagrams

AP Course Topics

Integrals:

- Application of integrals

NCTM Principles and Standards

- Algebra
- Geometry
- Problem Solving
- Connections

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presence in the solar system, we can look forward to far-off visions of the past becoming realities of the future.

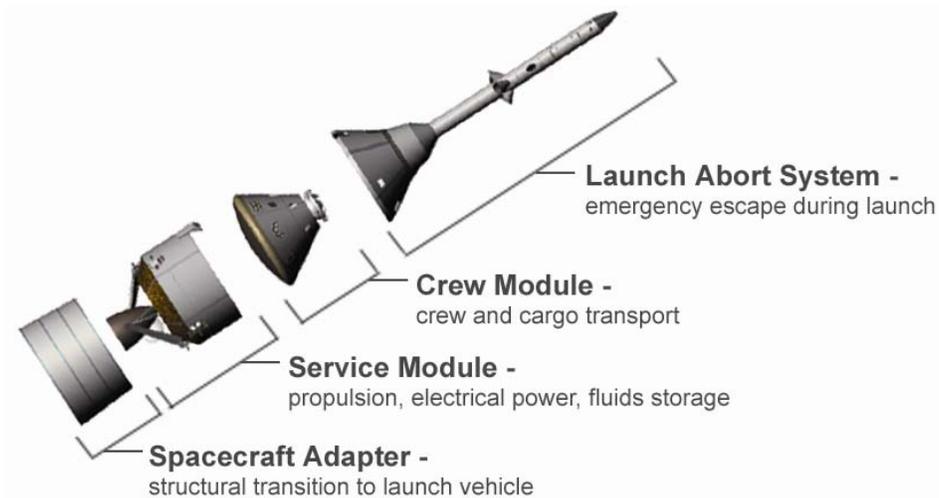


Figure 1: Components of the Orion spacecraft (NASA concept)

NASA is considering the Orion as the next space craft to carry a new generation of explorers into space. Orion may also be used as a crew return vehicle for astronauts coming home from the International Space Station (ISS) following the retirement of the space shuttle. Figure 1 shows some components of the Orion spacecraft. Orion would use an improved, larger blunt-body capsule, much like the shape of the Apollo capsule (Figure 2). In its current design with an outside diameter of 5 meters, the Orion crew module exceeds the Apollo capsule with three times the volume.

During Orion’s planning process, NASA engineers and scientists studied several different kinds of entry vehicles and rockets. Although Apollo-era researchers were consulted, NASA did not set out to make the Orion spacecraft identical to the Apollo spacecraft. Ultimately, a capsule design was found to meet the requirements while being the most effective within the safety goals.



NASA concept of the Orion crew module



Apollo capsule

Figure 2: Shape comparison of the Orion crew module and the Apollo capsule (not to scale)



AP Course Topics

Integrals

- Application of integrals.

NCTM Principles and Standards

Algebra

- Analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior.
- Write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency – mentally or with paper and pencil in simple cases and using technology in all cases.
- Judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
- Draw reasonable conclusions about a situation being modeled.

Geometry

- Visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections.

Problem Solving

- Build new mathematical knowledge through problem solving.
- Solve problems that arise in mathematics and in other contexts.
- Apply and adapt a variety of appropriate strategies to solve problems.
- Monitor and reflect on the process of mathematical problem solving.

Connections

- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.



Problem

As a member of the Orion engineering design team, it is your job to use the provided Computer Aided Design (CAD) drawing (NASA concept of the Orion crew module) to answer the following questions.

- Find the volume, in cubic meters, of the Orion crew module (CM). Make sure you use the correct significant digits.
- Within the outer shell of the CM will be a crew cabin that is pressurized (Figure 3). Suppose the pressurized crew cabin is approximately 55% of the total volume. What is the volume of the pressurized crew cabin? Please round your answer to the nearest cubic meter.

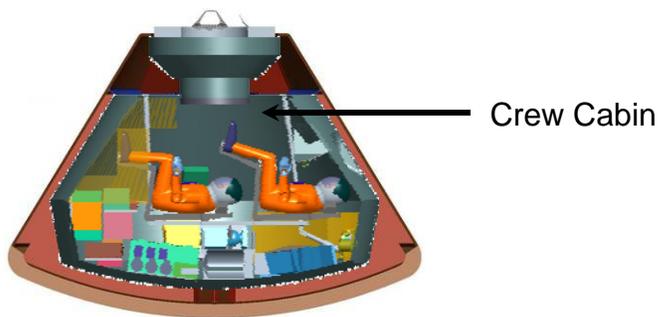
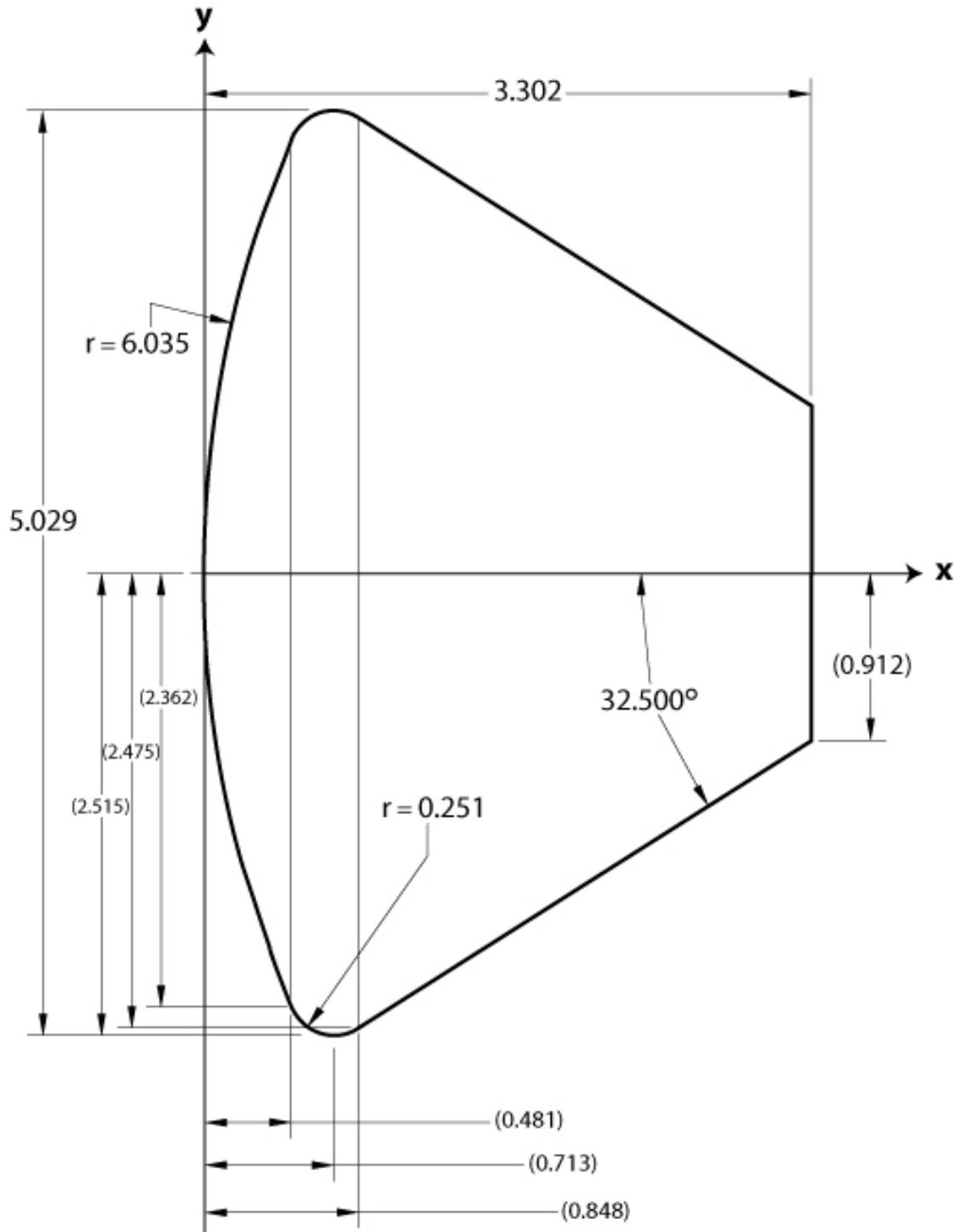


Figure 3: Cross-section of the Orion CM (NASA concept)

- Compare your results to the rooms in your home. Is there a room in your home that has about the same volume as the crew cabin of the CM? Explain.
- Using the information provided in the Background section, estimate the total volume of the Apollo capsule. Please round your answer to the nearest cubic meter.



NASA Concept of the Orion Crew Module
CAD Drawing



All measurements in meters unless otherwise noted.

**Solution Key** (One Approach)

- A. Find the volume, in cubic meters, of the Orion crew module (CM). Make sure you use the correct significant digits.

Note: The shaded region is a piecewise function consisting of more than one function.

Step 1: Identify all points of interest needed to determine $y=f(x)$ for each function.

Function #1: (0, 0), (0.481, 2.362)

Function #2: (0.481, 2.362), (0.713, 2.515), (0.848, 2.475)

Function #3: (0.848, 2.475), (3.302, 0.912)

Step 2: Find $y=f(x)$ for each individual function.**Function #1:**

Points of Interest: (0, 0), (0.481, 2.362)

Radius: $r = 6.035$ m

Center: $(h, k) = (6.035, 0)$

Standard form of the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 6.035)^2 + (y - 0)^2 = (6.035)^2$$

$$y^2 = (6.035)^2 - (x - 6.035)^2$$

$$y = \sqrt{(6.035)^2 - (x - 6.035)^2}$$

Function #2:

Points of interest: (0.481, 2.362), (0.713, 2.515), (0.848, 2.475)

Radius: $r = 0.251$ m

General form of the equation of a circle:

$$x^2 + y^2 + Bx + Cy + D = 0$$



Point (0.481, 2.362)

$$(0.481)^2 + (2.362)^2 + 0.481B + 2.362C + D = 0$$

$$0.231 + 5.579 + 0.481B + 2.362C + D = 0$$

$$0.481B + 2.362C + D = -5.810 \quad (1)$$

Point (0.713, 2.515)

$$(0.713)^2 + (2.515)^2 + 0.713B + 2.515C + D = 0$$

$$0.508 + 6.325 + 0.713B + 2.515C + D = 0$$

$$0.713B + 2.515C + D = -6.833 \quad (2)$$

Point (0.848, 2.475)

$$(0.848)^2 + (2.475)^2 + 0.848B + 2.475C + D = 0$$

$$0.719 + 6.126 + 0.848B + 2.475C + D = 0$$

$$0.848B + 2.475C + D = -6.845 \quad (3)$$

Solve for coefficients B , C , and D

$$(1) \quad 0.481B + 2.362C + D = -5.810$$

$$(2) \quad 0.713B + 2.515C + D = -6.833$$

$$-0.232B - 0.153C = 1.023 \quad (4)$$

$$(2) \quad 0.713B + 2.515C + D = -6.833$$

$$(3) \quad 0.848B + 2.475C + D = -6.845$$

$$-0.135B + 0.040C = 0.012 \quad (5)$$

$$(4) \quad -0.232B - 0.153C = 1.023$$

$$(5) \quad -0.135B + 0.040C = 0.012$$



Multiply (4) by 0.135 and (5) by -0.232

$$-0.031B - 0.021C = 0.138$$

$$0.031B - 0.009C = -0.003$$

$$-0.030C = 0.135$$

$$C = -4.500$$

$$-0.135B + 0.040(-4.500) = 0.012$$

$$-0.135B - 0.180 = 0.012$$

$$-0.135B = 0.192$$

(5)

$$B = -1.422$$

$$0.713(-1.422) + 2.515(-4.500) + D = -6.833$$

$$-1.014 - 11.318 + D = -6.833$$

(2)

$$D = 5.499$$

Coefficients

$$B = -1.422$$

$$C = -4.500$$

$$D = 5.499$$

General equation of the circle for function #2:

$$x^2 + y^2 - 1.422x - 4.500y = -5.499$$

Find the standard form of the equation of the circle by completing the square.

$$x^2 + y^2 - 1.422x - 4.500y = -5.499$$

$$(x^2 - 1.422x) + (y^2 - 4.500y) = -5.499$$

$$(x^2 - 1.422x + 0.506) + (y^2 - 4.500y + 5.063) = -5.499 + 0.506 + 5.063$$

$$(x - 0.711)^2 + (y - 2.250)^2 = 0.070$$

$$(y - 2.250)^2 = 0.070 - (x - 0.711)^2$$

$$y - 2.250 = \sqrt{0.070 - (x - 0.711)^2}$$

$$y = \sqrt{0.070 - (x - 0.711)^2} + 2.250$$



Function #3:

Points of interest: (0.848, 2.475), (3.302, 0.912)

Find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{0.912 - 2.475}{3.302 - 0.848}$$

$$m = \frac{-1.563}{2.454}$$

$$m = -0.637$$

Equation of a line in point-slope form:

$$y - y_1 = m(x - x_1)$$

$$y - 2.475 = -0.637(x - 0.848)$$

$$y = -0.637x + 0.540 + 2.475$$

$$y = -0.637x + 3.015$$

Piecewise Function

$$y = \left\{ \begin{array}{ll} \sqrt{(6.035)^2 - (x - 6.035)^2} & (0 \leq x < 0.481) \\ \sqrt{0.070 - (x - 0.711)^2} + 2.250 & 0.481 \leq x < 0.848 \\ -0.637x + 3.015 & 0.848 \leq x < 3.302 \end{array} \right\}$$

Step 3: Find the volume of the CM (solid), in cubic meters, by revolving the shaded region, R, about the x-axis.

Application of integration: Disc Method

$$\text{Volume} = V = \pi \int_a^b [R(x)]^2 dx$$

$$V = \pi \int_0^{0.481} \left[\sqrt{(6.035)^2 - (x - 6.035)^2} \right]^2 dx + \pi \int_{0.481}^{0.848} \left[\sqrt{0.070 - (x - 0.711)^2} + 2.250 \right]^2 dx + \pi \int_{0.848}^{3.302} [-0.637x + 3.015]^2 dx$$



$$V = \pi(1.359) + \pi(2.268) + \pi(7.535) \quad (\text{Used graphing calculator to evaluate the integral})$$

$$V = 35.066 \text{ m}^3$$

- B. Within the outer shell of the CM will be a crew cabin that is pressurized (Figure 3). Suppose the pressurized crew cabin is approximately 55% of the total volume. What is the volume of the pressurized crew cabin? Please round your answer to the nearest cubic meter.

$$V_p = \text{Pressurized Volume}$$

$$V_t = \text{Total Volume}$$

$$V_p = V_t \cdot 0.55$$

$$V_p = (35.066 \text{ m}^3) \cdot 0.55$$

$$V_p = 19 \text{ m}^3$$

- C. Compare the results to the rooms in your home. Is there a room in your home that has about the same volume as the crew cabin of the CM? Explain.

Answers will vary. A common bedroom in a house is approximately 3.7 m (12 ft) x 3.4 m (11 ft) with a 2.4 m (8 ft) ceiling. The total volume of a room that size is 30.2 m^3 (1056 ft^3)

- D. Using the information provided in the Background section, estimate the total volume of the Apollo capsule. Please round your answer to the nearest cubic meter.

$$V_a = \text{Volume of Apollo capsule}$$

$$V_t = \text{Volume of Orion crew module}$$

Orion will have approximately 3 times the volume of an Apollo capsule.

$$V_a = \frac{V_t}{3}$$

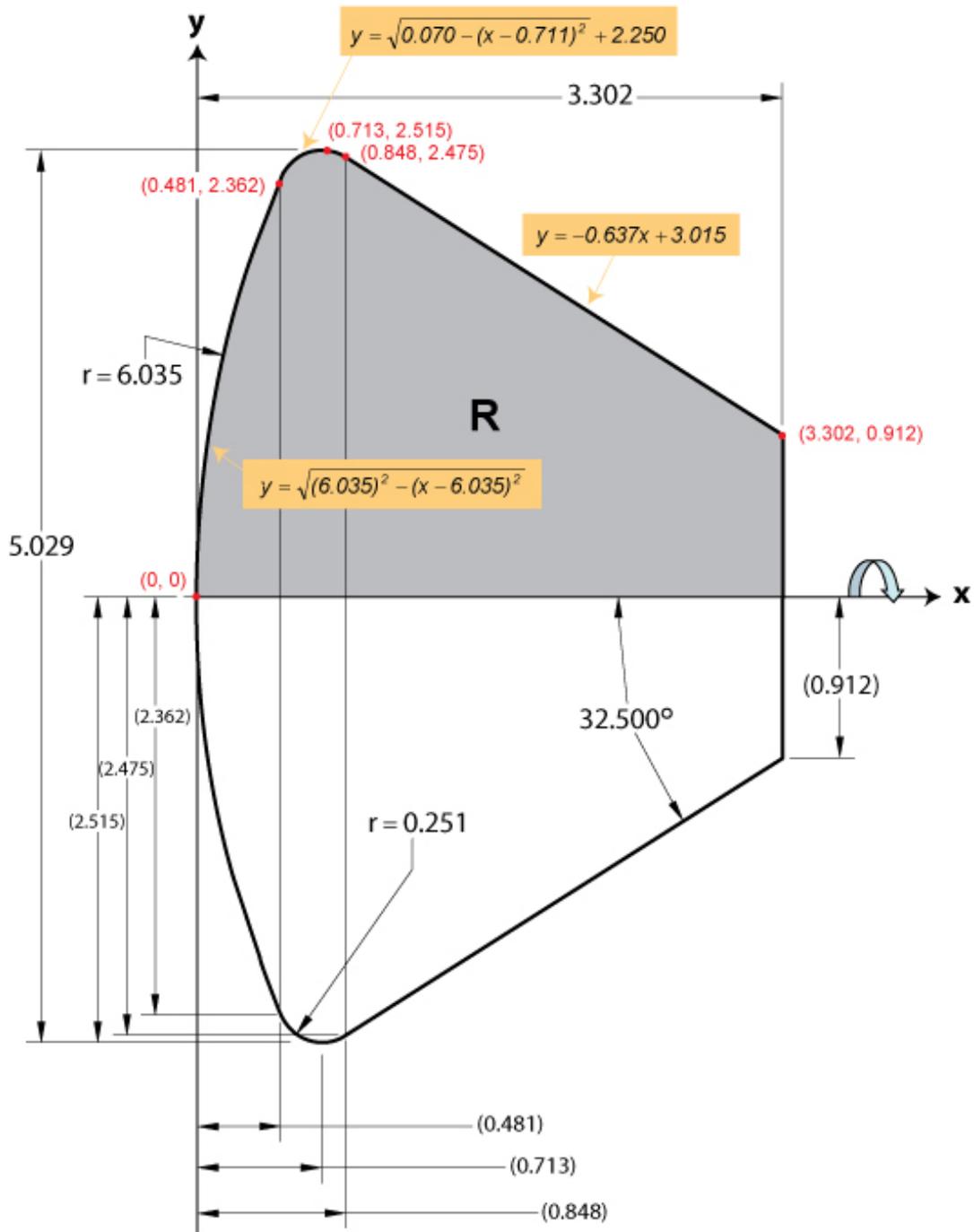
$$V_a = \frac{35.066 \text{ m}^3}{3}$$

$$V_a = 12 \text{ m}^3$$



Solution Key

NASA Concept of the Orion Crew Module
CAD Drawing



All measurements in meters unless otherwise noted.



Scoring Guide

Suggested 21 points total to be given if students find their own equations for Functions 1-3.

Suggested 14 points total if students use the given equations for Functions 1-3.

Question	Distribution of points
A <i>Function 1:</i> 2 points	1 point - standard equation of circle, 1 point - solves for y
<i>Function 2:</i> 4 points	1 point – system of equations 1 point solves system 1 point – general equation 1 point solves for y
<i>Function 3:</i> 1 point	1 point – equation of line
<i>Piecewise</i> <i>Function</i> 3 points	1 point for each line with correct domain restriction
<i>Volume</i> <i>Calculation</i> 7 points	6 points for the integral expression – 2 for each piece of the piecewise function with correct limits of integration. 1 point for the correct answer
B 1 point	1 point for the correct answer
C 2 points	1 point for considering a typical room 1 point for comparing with the crew cabin
D 1 point	1 point for the correct answer



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Thanks to the subject matter experts for their contributions in developing this problem:

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