Quantum Metrology

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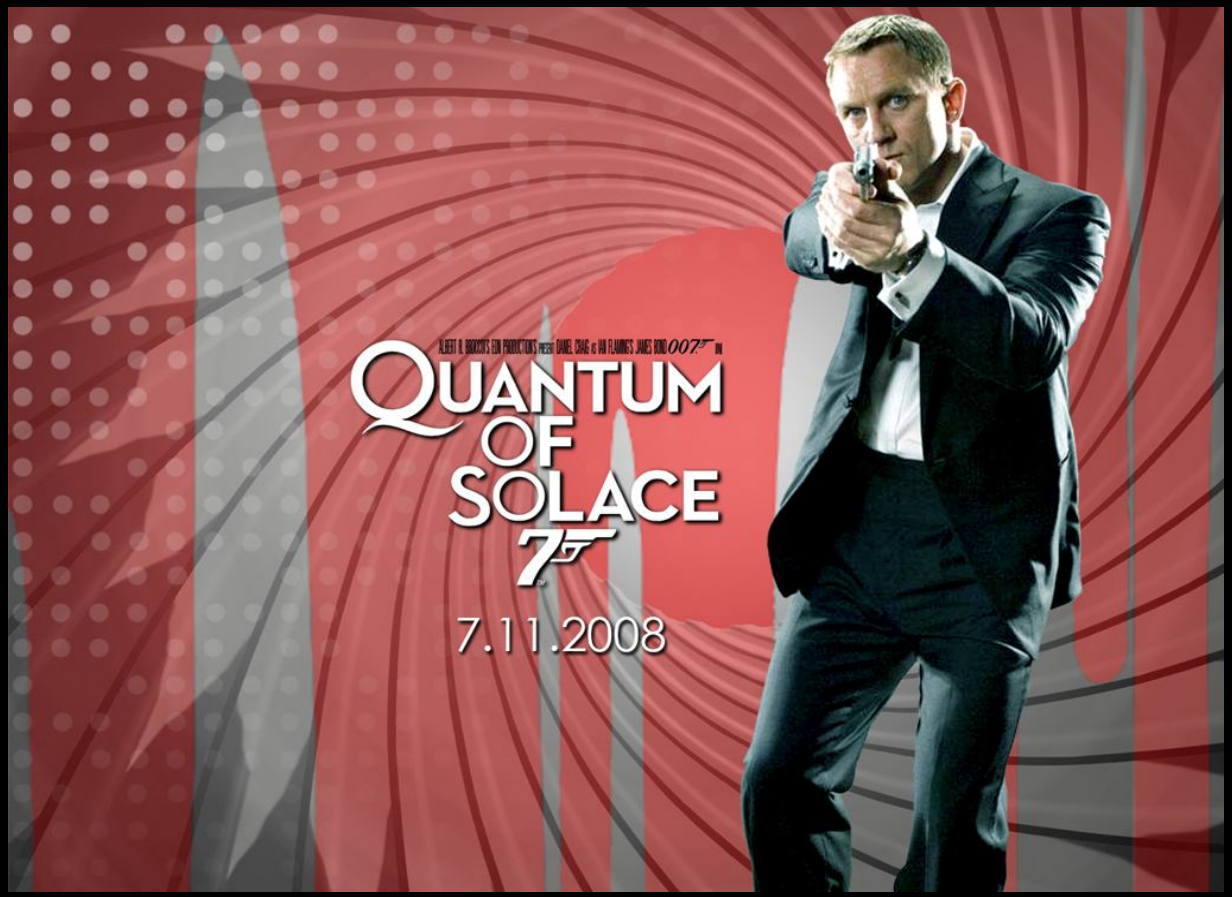


Outline

- Surprise and Relative Entropy
- Quantum Metrology, Mach Zehnder Interferometer
- Parameter Estimation Concept
- Some Previous Approaches
- Distinguishability and Δ
- Local Tracking vs Global Acquisition
- Local Distinguishability $\rightarrow$ Fisher Information
- Advantages of our Approach

Surprise and Uncertainty

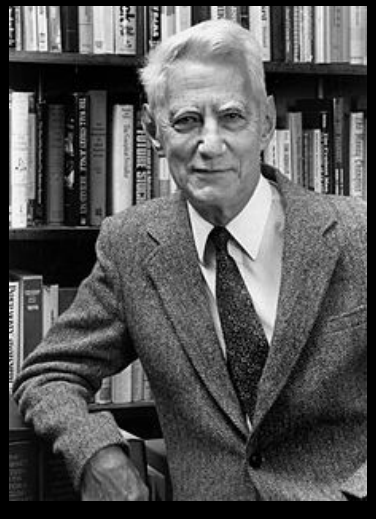
- Probability ↓, surprise ↑ and vice versa. Choose 1/P?
- Surprise is additive: Indep events 1,2: P = P1xP2 but combined surprise should be sum. Use Logs.
- Surprise = Log (1/P) = Log(1/P1)+Log(1/P2) = Log 1/ (P1 x P2)
- Average Surprise for a Distribution is 'Uncertainty':

$$\langle \mathrm{Log}\,(1/p) \rangle = \sum_{p} p\,\mathrm{Log}\,(1/p)$$

Uncertainy is known as

Shannon Entropy,

after Claude Shannon, 1916-2001.



Relative Entropy

- What if we expect our data to belong to a probability distribution Q(X), yet the data set is typical of distribution P(X) for random variable X?

- Average expected Surprise is

$$\langle \mathrm{Log}[1/Q(x)] \rangle = \sum_{x \in X} P(x)\mathrm{Log}[1/Q(x)]$$

- Actual Surprise is

$$\langle \mathrm{Log}[1/P(x)] \rangle = \sum_{x \in X} P(x)\mathrm{Log}[1/P(x)]$$

- Surprise discrepancy is

$$S[P \,||\, Q] = \sum_{x \in X} P(x)\mathrm{Log}[P(x)/Q(x)]$$

- This is the <u>Relative Entropy</u> or Kullbach-Leibler divergence.

- Not a metric distance: S[P||Q] ≠S[Q||P], but non-negative. Related to metric distance as P approaches Q.

$$d_I(\theta, \theta + \epsilon) \approx \sqrt{2}\,(S[\,p(x|\theta)||p(x|\theta + \epsilon)\,])^{\frac{1}{2}}$$

- In Bayes experiments one tries to maximize Relative Entropy between Posterior and Prior distributions.

Relative Entropy in Use: Coin Toss

Coin Toss Sequence: HHHTHHTHHHTHTHHTHHTT

You decide: Fair or Unfair Coin? Hx7, Tx13

What is the likelihood the coin is fair, i.e. that governing distribution is fair: Pf(H)=Pf(T)=1/2?

Data distribution is Pd(H)=7/20, Pd(T)=13/20.

$$P_{F\to D} \le 2^{-kS}$$

Upper bound on probability that the coin is fair:

P < 2^{-k S(Pf||Pd)}, where k = 20 is the number of tosses, and S[Pf||Pd] is the Relative Entropy distance between the data distribution and the fair distribution.

Calculation gives P < 2^{-20 * (0.066) } = 0.401, i.e. Coin has less than 40% chance of being fair.

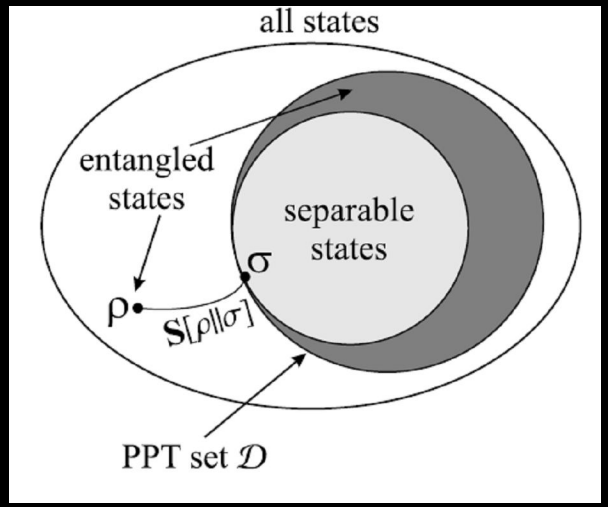
Relative Entropy in Quantum Mechanics

- Quantum Relative Entropy is measured between density matrices rather than probability distributions:

$$S[\rho \,\|\, \sigma] = \rho \mathrm{Log}(\rho) - \rho \mathrm{Log}(\sigma)$$

- Relative Entropy of Entanglement is Relative Entropy between an Entangled State and 'nearest' state in the Separable Set D.

$$E_R[\rho] = \inf_{\sigma \in D} S[\rho \,\|\, \sigma]$$



PHYSICAL REVIEW A **70**, 062305 (2004)

Resilience of multiphoton entanglement under losses

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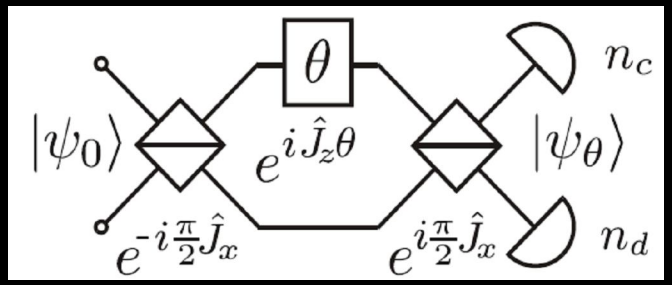
We analyze the resilience under photon loss of the bipartite entanglement present in multiphoton states produced by parametric down-conversion. The quantification of the entanglement is made possible by a symmetry of the states that persists even under polarization-independent losses. We examine the approach of the states to the set of positive partial transpose states as losses increase, and calculate the relative entropy of entanglement. We find that some bipartite distillable entanglement persists for arbitrarily high losses.

Interferometry & Metrology

- Prototype Mach Zehnder performs rotation about y-axis.



Schrödinger equation	$i\frac{\partial}{\partial\theta}\lvert\psi_\theta\rangle = \hat{J}_y\lvert\psi_\theta\rangle$
evolution	$e^{i\hat{J}_y\theta}\lvert\psi_0\rangle = \lvert\psi_\theta\rangle$

- <u>Interferometry → Classical Parameter Estimation</u>: We know Hamiltonian = Jy, but what is rotation angle θ?
- <u>Metrology</u>: How precise is any unbiased estimate we make of θ?
- P(M|θ) typically multiple-peaked and periodic in θ, take care when using conventional indicators, e.g. variance, moments, maximum likelihood approach.
- <u>Schwinger Spin Representation</u>: (nc+nd)=2j, (nc-nd)=2m

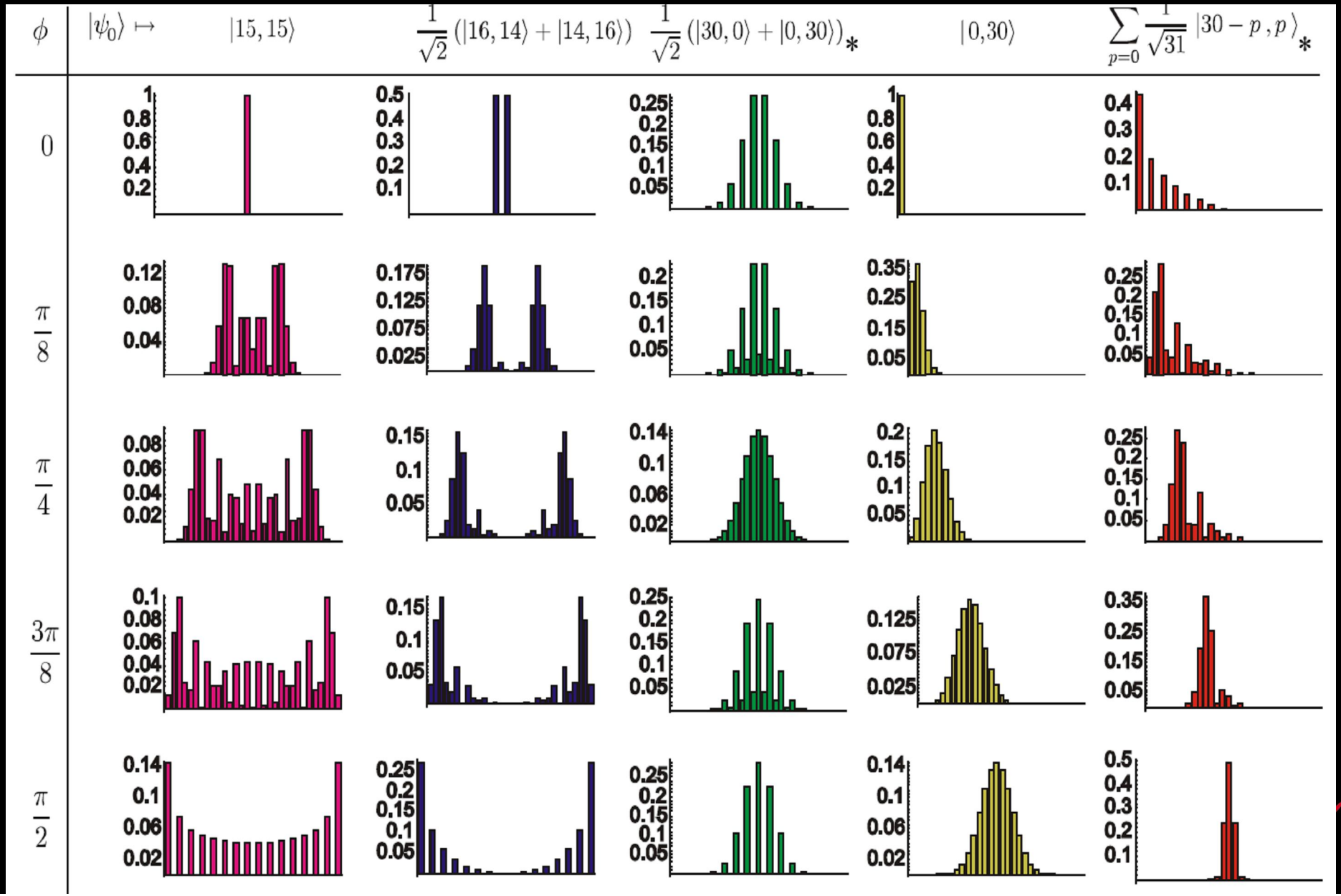
$$\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)/2, \quad \hat{J}_y = (\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger)/2i,$$
$$\hat{J}_z = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2, \quad \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2,$$

Accuracy and Precision

High Accuracy
Low Precision

High Precision
Low Accuracy

Photon Counting Distribution of MZ



How has the problem been tackled?
(more than one way to skin Schrodinger's Cat)

- Error Propagation.
- Bahder & Lopata: 'Fidelity'
- Pezze & Smerzi: Simple Bayesian Analysis
- Berry & Wiseman: Sharpness & Holevo Variance

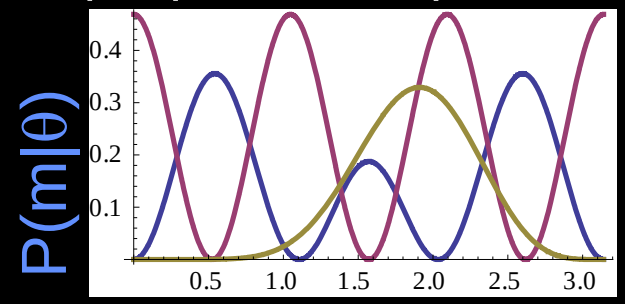
Error Propagation

- Most popular approach in the literature.
- Simple to grasp, easy to calculate.

$$\delta\phi = \frac{\Delta\hat{A}_N}{\left|\partial\langle\hat{A}_N\rangle/\partial\phi\right|}$$

$$\Delta\hat{A}_N = \sqrt{\langle\hat{A}_N^2\rangle - \langle\hat{A}_N\rangle^2}$$

- What about multiple peaks and periodic distributions? No applicability.



Graph for nc=4, nd=2 detected

- Inconclusive or incorrect scaling for pathological probe states, e.g. those that are symmetric have $\langle\hat{J}_z\rangle = 0$ for all phases ϕ. But even a simple input $|\mathrm{j},\mathrm{m}\rangle$ gives scaling:

$$\sqrt{\frac{2m^2}{j(j+1)-m^2}}$$

This should be contrasted with the true scaling, from Cramer Rao:

$$\sqrt{\frac{1}{2j(j+1)-2m^2}}$$

Badher & Lopata: Fidelity of Qm Interferometers

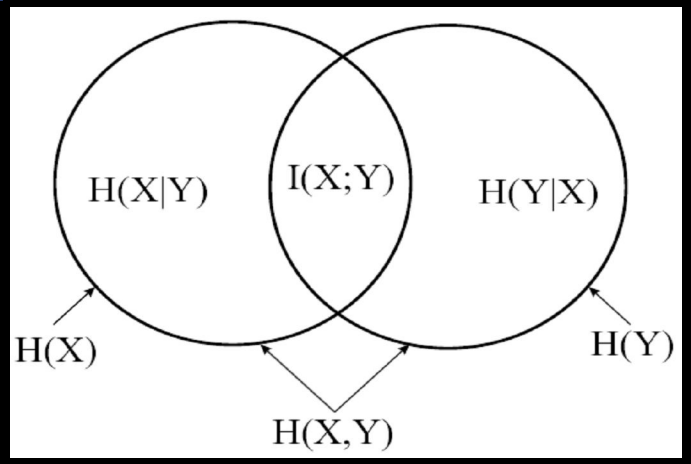
▪Phys. Rev. A 74 051801 (2006)

•Acknowledge problems with Error Propagation model.

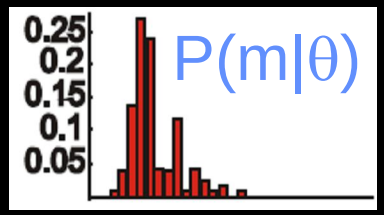
•Propose Mutual Information Between posterior Phase Distribution P(m|θ) and Phase θ.

$$I_M(X;Y) = S[P(X,Y) \,\|\, P(X)P(Y)]$$

It is uncertainty reduction in P(θ) due to knowledge of P(m).



•Want a strong correlation between phase and measurement, but is this enough to quantify confidence in inference:



→ θ?

•Also, what about quantifying the uncertainty of P(θ)? This should also be minimized.

Pezze & Smerzi

▪Phys. Rev. A 73 011801 (R) (2006)

- Also criticize linear error propagation approach
- Promote a Bayesian Analysis for event A followed in time by B.

  $P(A|B) = P(B|A) \times P(A) \;/\; P(B)$, where $P(B) = \sum_A P(B|A) \times P(A)$

  (posterior) (prior)

- They analyse probe states of form $\frac{(|j,+m\rangle + |j,-m\rangle)}{\sqrt{2}}$
- Analysis limited to $\theta = 0$, i.e. when the interferometer doesn't change the input state!
- Conclusion is that m=1 in states above represents best state for interferometry, with no caveats.
- However in our calculation of Cramer-Rao bound: $p_m(\theta) = |\,{}_z\langle j,m|\psi_\theta\rangle|^2$

$$(\delta\phi_e)^2 \geq 1/\mathcal{J}(\theta)$$

$$\mathcal{J}(\theta) = \sum_{m=-j}^{+j} p_m(\theta)\left(\frac{\partial}{\partial\theta}\ln p_m(\theta)\right)^2 = 2[j(j+1) - m^2]$$

- Bayesian Calculation has exp high computational overhead.

Berry & Wiseman

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Optimal States and Almost Optimal Adaptive Measurements for Quantum Interferometry

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(Received 19 June 2000)

Holevo phase variance [10]

$$(\Delta\phi)^2 = V_\phi \equiv S_\phi^{-2} - 1\,, \tag{6}$$

where $S_\phi \in [0, 1]$ is the *sharpness* of the phase distribution, defined as

$$S_\phi \equiv |\langle e^{i\phi}\rangle| = \int_0^{2\pi} d\phi\, P(\phi) e^{i(\phi-\bar{\phi})}, \tag{7}$$

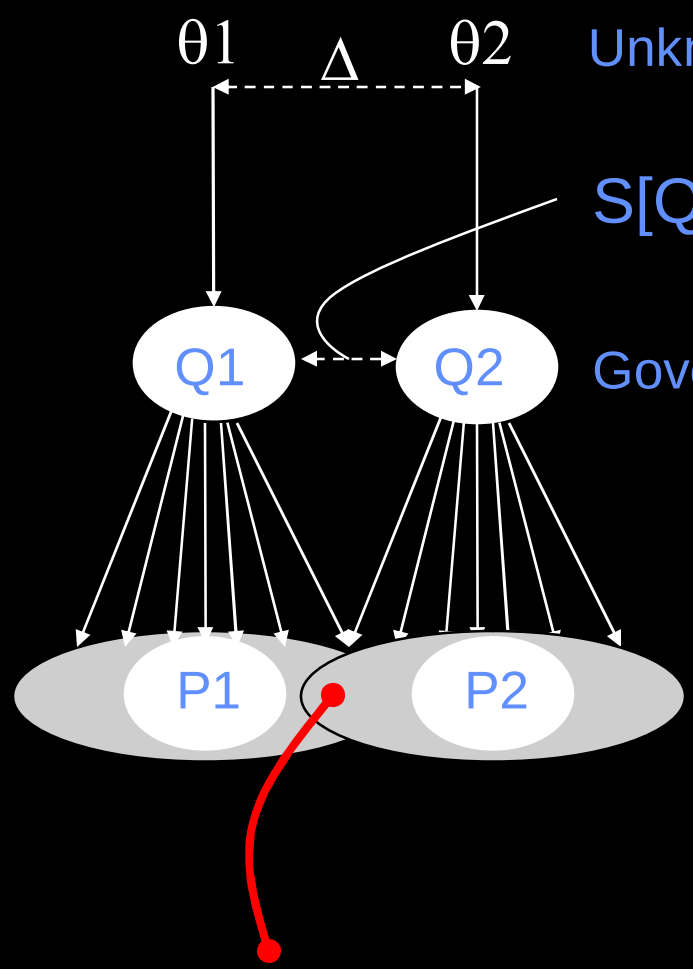
where the "mean phase" $\bar{\phi}$ is defined so that S_ϕ is positive. The Holevo variance is the natural quantifier for dispersion in a cyclic variable. If the variance is small, then it is easy to show that

$$V_\phi \simeq \int_0^{2\pi} 4\sin^2\left(\frac{\phi-\bar{\phi}}{2}\right) P(\phi)\, d\phi\,, \tag{8}$$

from which the equivalence to the usual definition of variance for well-localized distributions is readily apparent.

Drawback:Sharpness needs redefining depending on periodicity of P(ϕ), not a Universal methodology.

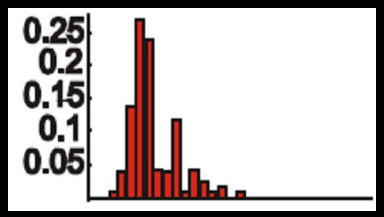
Schematic of Phase Inference



Unknown phase could be θ1 or θ2

S[Q1||Q2] → Maximize!

Governing Distribution Q



Distribution P of Measured Data: Typical is in White, Atypical is Grey

Overlap Region: Ambiguity in Inferring θ

Our Approach: Intrinsic Distinguishability

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Local and Global Distinguishability in Quantum Interferometry

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A statistical distinguishability based on relative entropy characterizes the fitness of quantum states for phase estimation. This criterion is employed in the context of a Mach-Zehnder interferometer and used to interpolate between two regimes of local and global phase distinguishability. The scaling of distinguishability in these regimes with photon number is explored for various quantum states. It emerges that local distinguishability is dependent on a discrepancy between quantum and classical rotational energy. Our analysis demonstrates that the Heisenberg limit is the true upper limit for local phase sensitivity. Only the "NOON" states share this bound, but other states exhibit a better trade-off when comparing local and global phase regimes.

Task: To unambiguously perform inverse map $P(\theta) \mapsto \theta$

We propose that a global *distinguishability* for estimating a phase θ, previously known to exist within a finite interval Δ centered on $\theta = \chi$ be defined as

$$\mathcal{D}(\chi, \Delta) = \frac{1}{\Delta^2} \int_{\theta_1, \theta_2 = \chi - \Delta/2}^{\chi + \Delta/2} S[P(\theta_1) \parallel P(\theta_2)] d\theta_1 d\theta_2.$$

$$S[P(\theta_1) \parallel P(\theta_2)] = \sum_{m=-j}^{+j} p_m(\theta_1) \log_2 \frac{p_m(\theta_1)}{p_m(\theta_2)}$$

New parameter Δ, *a priori* precision, determines global (large Δ) vs. local (small Δ) nature of Metrology Task.

Local and Global Parameter Estimation

- Contrast pre- and post- target locking.
- Initially target location is unknown, but exists within some large global $\Delta \sim 2\pi$ window.
- After initial lock, target is being tracked within a tight localized $\Delta \sim 0$ parameter range.
- Q: Are some probe states better for a global phase acquisition, and others for local tracking?

Local Distinguishability & Fisher Information

distinguishability $\mathcal{D}(\chi,\Delta)=\frac{1}{\Delta^2}\int_{\theta_1,\theta_2=\chi-\Delta/2}^{\chi+\Delta/2} S[P(\theta_1)||P(\theta_2)]d\theta_1 d\theta_2$

For vanishingly small $\Delta \sim 0$, $\mathcal{D}$ approaches a *local* distinguishability:

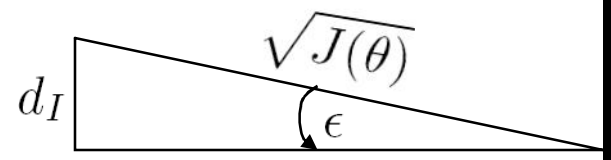
$$\frac{1}{4}\Big[S[P(\chi-\Delta/2)||P(\chi+\Delta/2)]+S[P(\chi+\Delta/2)||P(\chi-\Delta/2)]\Big]$$
$$=\frac{\Delta^2}{8\ln 2}\{\,\mathcal{J}(\chi-\Delta/2)+\mathcal{J}(\chi+\Delta/2)\,\}+O(\Delta^3)$$

Above, $\mathcal{J}(\theta)$ is the classical Fisher Information for the measurement distribution $P(\theta)$,

$$\mathcal{J}(\theta)=\sum_{m=-j}^{+j} p_m(\theta)\left(\frac{\partial}{\partial\theta}\ln p_m(\theta)\right)^2$$

Cramér-Rao bound $(\delta\phi_e)^2 \geq 1/\mathcal{J}(\theta)$

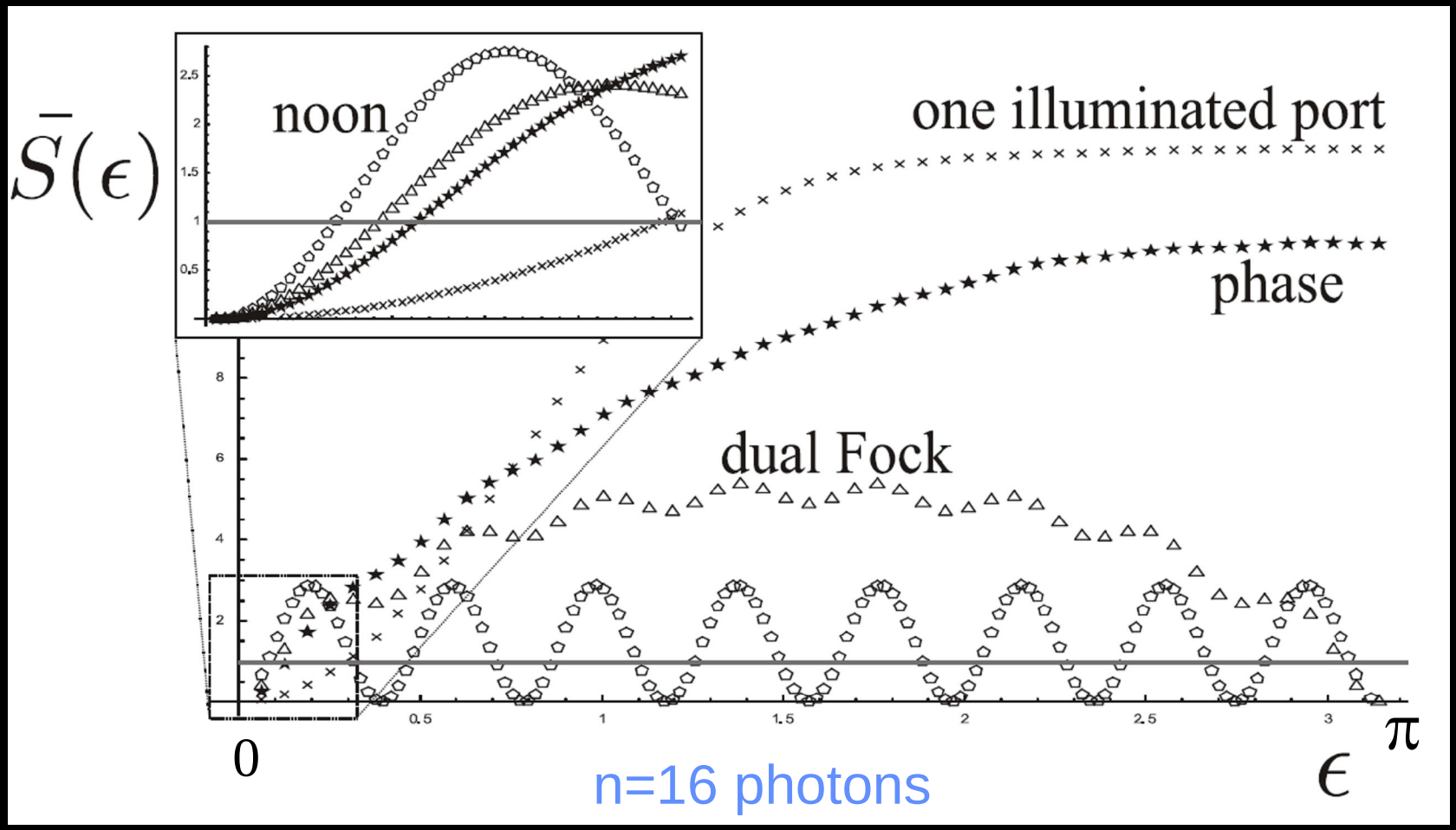
$$d_I(\theta,\theta+\epsilon)\approx\sqrt{J(\theta)}\epsilon$$



Here is a table of Fisher Information for various probe states

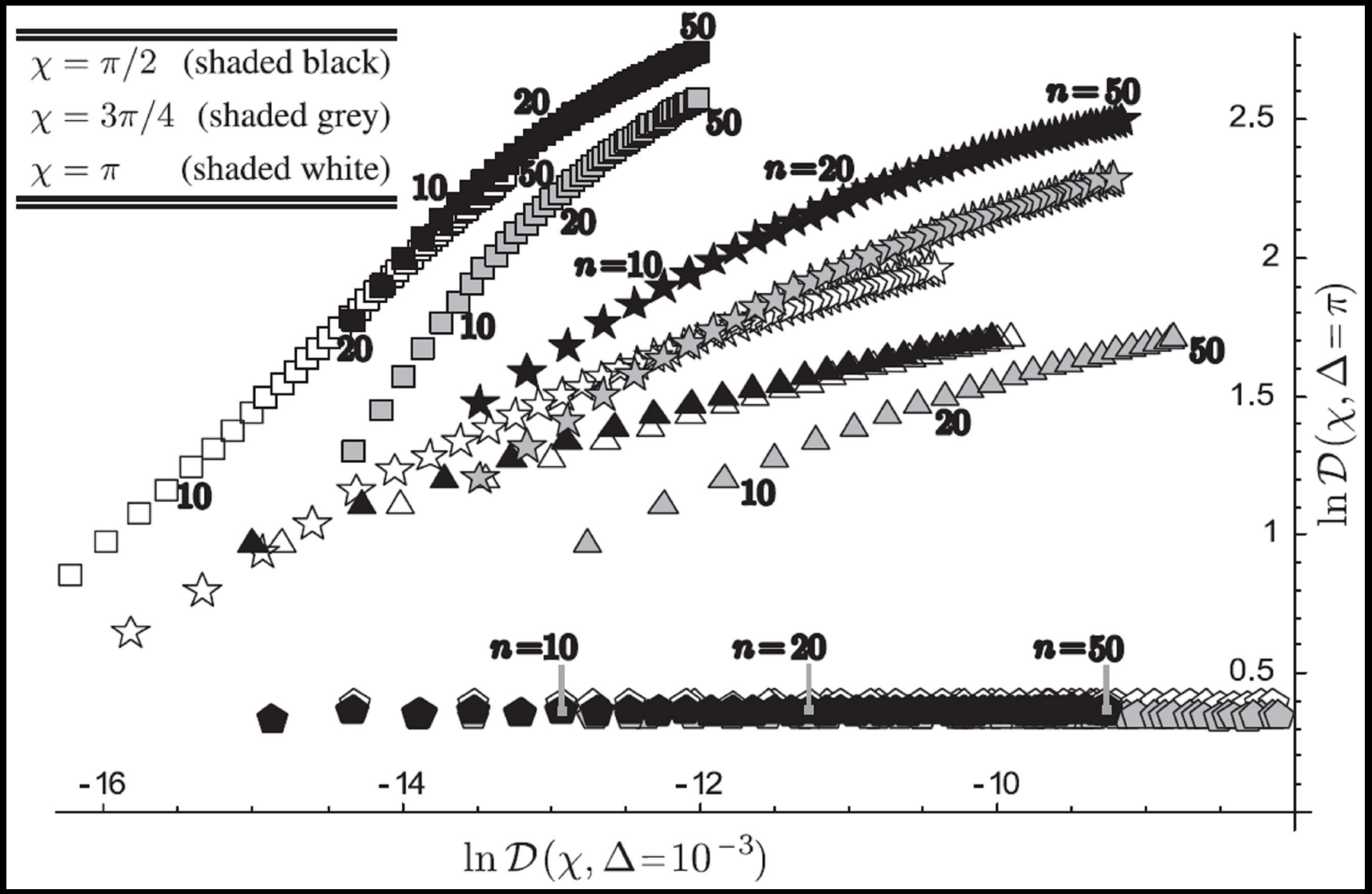
$\lvert\psi_0\rangle$	$\mathcal{J}(\theta)$
$\lvert j,+j\rangle_z$	$2j$
$\lvert j,0\rangle_z$	$2j(j+1)$
$(\lvert j,+j\rangle_y+e^{i\zeta}\lvert j,-j\rangle_y)/\sqrt{2}$	$4j^2$
$(\lvert j,+m\rangle_y+e^{i\zeta}\lvert j,-m\rangle_y)/\sqrt{2}$	$4m^2$
$\lvert j,m\rangle_z$	$2[j(j+1)-m^2]$
$\lvert j,\gamma\rangle=\frac{1}{\sqrt{2j+1}}\sum_{m=-j}^{j}e^{im\gamma}\lvert j,m\rangle_y$	$\frac{4}{3}j(j+1)$

Our Results I



$$\bar{S}(\epsilon) = \int_{2\pi} d\theta \{ S[\theta || \theta + \epsilon] + S[\theta + \epsilon || \theta] \}$$

Our Results II



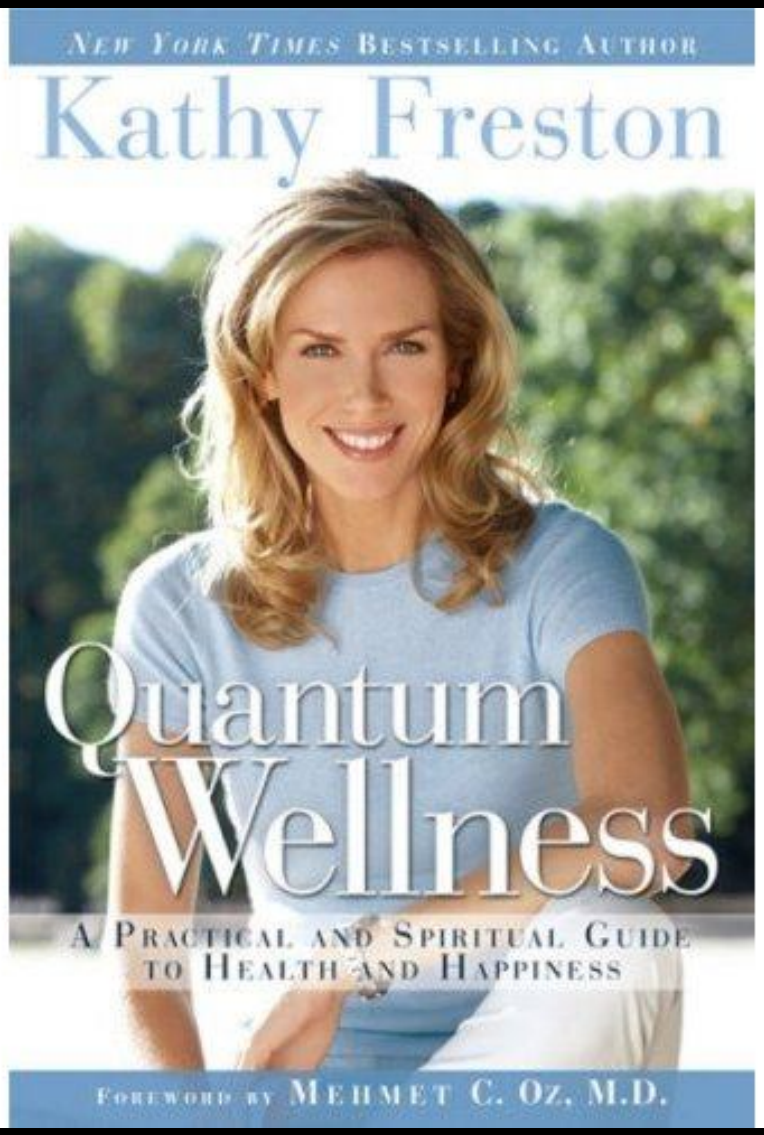
Probe States: NOON (pentagons), Phase States (stars), Twin Fock (triangles), Coherent Light+Vacuum (squares)

Advantages to our Approach

- Distinguishability is property of whole distribution, not just partial info, such as moments provide.
- We examine Photon-Counting Measurements, but analysis applies to any type, including quadratures and POVMs.
- The lower bounds we provide to sensitivity are rigorous and require no approximation such as

  $n \sim \infty$, or $\theta \sim 0$.
- Δ is a new sub-wavelength parameter, providing a local/global context to the measurement task, and an indication of what probe states to use.

What Next? The Future…

- Add Noise and Decoherence.
- Apply to other measurement types, e.g. Homodyning and POVMs.
- Use this rigorous universal methodology to move beyond intrinsic fitness of states, to develop a Phase Estimation Protocol.
- Adapt for Atom Interferometry and BECs.

Thank You for Listening